Deciding Relaxed Two-Colorability—A Hardness Jump

Robert Berke and Tibor Szabó

Institute of Theoretical Computer Science, ETH Zürich, 8092 Switzerland
{berker, szabo}@inf.ethz.ch

Abstract. A coloring is proper if each color class induces connected components of order one (where the order of a graph is its number of vertices). Here we study relaxations of proper two-colorings, such that the order of the induced monochromatic components in one (or both) of the color classes is bounded by a constant. In a \((C_1, C_2)\)-relaxed coloring of a graph \(G\) every monochromatic component induced by vertices of the first (second) color is of order at most \(C_1\) (\(C_2\), resp.). We are mostly concerned with \((1, C)\)-relaxed colorings, in other words when/how is it possible to break up a graph into small components with the removal of an independent set.

We prove that every graph of maximum degree at most three can be \((1, 22)\)-relaxed colored and we give a quasilinear algorithm which constructs such a coloring. We also show that a similar statement cannot be true for graphs of maximum degree at most 4 in a very strong sense: we construct 4-regular graphs such that the removal of any independent set leaves a connected component whose order is linear in the number of vertices.

Furthermore we investigate the complexity of the decision problem \((\Delta, C)\)-AsymRelCol: Given a graph of maximum degree at most \(\Delta\), is there a \((1, C)\)-relaxed coloring of \(G\)? We find a remarkable hardness jump in the behavior of this problem. We note that there is not even an obvious monotonicity in the hardness of the problem as \(C\) grows, i.e. the hardness for component order \(C + 1\) does not imply directly the hardness for \(C\). In fact for \(C = 1\) the problem is obviously polynomial-time decidable, while it is shown that it is NP-hard for \(C = 2\) and \(\Delta \geq 3\).

For arbitrary \(\Delta \geq 2\) we still establish the monotonicity of hardness of \((\Delta, C)\)-AsymRelCol on the interval \(2 \leq C \leq \infty\) in the following strong sense. There exists a critical component order \(f(\Delta) \in \mathbb{N} \cup \{\infty\}\) such that the problem of deciding \((1, C)\)-relaxed colorability of graphs of maximum degree at most \(\Delta\) is NP-complete for every \(2 \leq C < f(\Delta)\), while deciding \((1, f(\Delta))\)-colorability is trivial: every graph of maximum degree \(\Delta\) is \((1, f(\Delta))\)-colorable. For \(\Delta = 3\) the existence of this threshold is shown despite the fact that we do not know its precise value, only \(6 \leq f(3) \leq 22\). For any \(\Delta \geq 4\), \((\Delta, C)\)-AsymRelCol is NP-complete for arbitrary \(C \geq 2\), so \(f(\Delta) = \infty\).

We also study the symmetric version of the relaxed coloring problem, and make the first steps towards establishing a similar hardness jump.

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1 Introduction

A function from the vertex set of a graph to a $k$-element set is called a $k$-coloring. The values of the function are referred to as colors. A coloring is called proper if the value of the function differs on any pair of adjacent vertices. Proper coloring and the chromatic number of graphs (the smallest number of colors which allow a proper coloring) are among the most important concepts of graph theory. Numerous problems of pure mathematics and theoretical computer science require the study of proper colorings and even more real-life problems require the calculation or at least an estimation of the chromatic number. Nevertheless, there is the discouraging fact that the calculation of the chromatic number of a graph or the task of finding an optimal proper coloring are both intractable problems, even fast approximation is probably not possible. This is one of our motivations to study relaxations of proper coloring, because in some theoretical or practical situations a small deviation from proper is still acceptable, while the problem could become tractable. Another reason for the introduction of relaxed colorings is that in certain problems the use of the full strength of proper coloring is an “overkill”. Often a weaker concept suffices and provides better overall results.

In this paper we study various relaxations of proper coloring, which allow the presence of some small level of conflicts in the color assignment. Namely, we will allow vertices of one or more color classes to participate in one conflict or, more generally, let each conflicting connected component have at most $C$ vertices, where $C$ is a fixed integer, not depending on the order of the graph. Most of our results deal with the case of relaxed two-colorings.

To formalize our problem precisely we say that a two-coloring of a graph is $(C_1, C_2)$-relaxed if every monochromatic component induced by the vertices of the first color is of order at most $C_1$, while every monochromatic component induced by the vertices of the second color is of order at most $C_2$. Note that $(1, 1)$-relaxed coloring corresponds to proper two-coloring.

In the present paper we deal with the two most natural cases of relaxed two-colorings. We say symmetric relaxed coloring when $C_1 = C_2$ and asymmetric relaxed coloring when $C_1 = 1$. Symmetric relaxed colorings were first studied by Alon, Ding, Oporowski and Vertigan [1] and implicitly, even earlier, by Thomassen [18] who resolved the problem for the line graph of 3-regular graphs initiated by Akiyama and Chvátal [2]. Asymmetric relaxed colorings were introduced in [5].

**Related relaxations of proper colorings.** There are several other types of coloring concepts related to our relaxation of proper coloring.

In a series of papers Škrekovski [17], Havet and Sereni [8], and Havet, Kang, and Sereni [9] investigated the concept of improper colorings over various families of graphs. A coloring is called $(k, l)$-improper if none of the at most $k$ colors induces a monochromatic component containing vertices of degree larger than $l$. Hence in an improper coloring the amount of error is measured in terms of the maximum degree of monochromatic components rather than in terms of their order.