DI-DOMAINS AS A MODEL OF POLYMORPHISM
Thierry Coquand, Carl Gunter and Glynn Winskel
Computer Laboratory, University of Cambridge, Cambridge CB2 3QG, England

In this paper we investigate a model construction recently described by Jean Yves Girard. This model differs from the models of McCracken, Scott, etc. in that the types are interpreted (quite pleasingly) as domains rather than closures or finitary projections on a universal domain. Our objective in this paper is two-fold. First, we would like to generalize Girard’s construction to a larger category called dI-domains which was introduced by Berry [2]. The dI-domains possess many of the virtues of the domains used by Girard. Moreover, the dI-domains are closed under the separated sum and lifting operators from denotational semantics and this is not true of the domains of Girard. We intend to demonstrate that our generalized construction can be used to do denotational semantics in the ordinary way, but with the added feature of type polymorphism with a “types as domains” interpretation. Our second objective is to show how Girard’s construction (and our generalization) can be done abstractly. We also give a representational description of our own construction using the notion of a prime event structure.

1 Introduction

The polymorphic $\lambda$-calculus was discovered by Girard [6] and later rediscovered by Reynolds [14]. As was the case with the simple untyped $\lambda$-calculus, the syntax of the calculus was, at first, understood better than its semantics. A model for the polymorphic calculus was first presented by McCracken [10] based on the cpo of closures over the algebraic lattice of subsets of the natural numbers. A similar technique can be used [1] to build models for the polymorphic calculus using finitary projection models such as the ones described by Scott [15] and Gunter [8]. More recently still there has been progress in saying what a model of the polymorphic calculus is in general. As with the simple untyped calculus, this can be done through the use of environment models [3] or categorically [16].

In this paper we investigate a model construction recently described by Girard [7]. This model differs from the models of McCracken, Scott, etc. in that the types are interpreted (quite pleasingly) as domains rather than closures or finitary projections on a universal domain. The
construction is carried out over an interesting cartesian closed category of algebraic cpo's called *qualitative domains* which satisfy a very strong finiteness property. Our objective in this paper is two-fold. First, we would like to generalize Girard's construction to a larger category called *dI-domains* which was introduced by Berry [2]. The dI-domains possess many of the virtues of the qualitative domains. In addition, the dI-domains are closed under the separated sum and lifting operators from denotational semantics and this is not true of the qualitative domains. We intend to demonstrate that our generalized construction can be used to do denotational semantics in the ordinary way, but with the added feature of type polymorphism with the "types as domains" interpretation. For example, we will be able to interpret data types such as trees \((T \cong T + T)\) and S-expressions \((S \cong \text{Atoms} + (S \times S))\) in the way they are ordinarily interpreted in the Scott-Strachey theory. Other useful types based on the lift operation (such as the solution to the domain equation \(X \cong X_1\)) will also be available with our approach. As with the qualitative domains we will also be able to obtain solutions for equations (such as \(L \cong \text{Atoms} + L \rightarrow L\)) with higher types. Our second objective is to show how Girard's construction (and our generalization) can be done *abstractly*. An ultimate result might carry out these constructions for "qualitative categories" and "dI-categories". For the purposes of this paper, however, we will (usually) restrict ourselves to posets. We also give a *representational* description of our own construction using the notion of a *prime event structure* which was introduced by Nielsen, Plotkin and Winskel [11] and Winskel [18].

The paper is divided into four sections. In the second section we present background definitions for dI-domains, event structures, etc. and demonstrate some basic properties. The third section gives the basic model construction in abstract and representational styles. In the fourth section we discuss the calculus we seek to model which we call the *polymorphic fixedpoint calculus*.

We would like to accord significant credit to Jean-Yves Girard and Gérard Berry for the ideas of this paper. In fact, the idea of developing a theory which includes a separated sum was suggested by Girard in Annex B of [7] (although the specific choice of dI-domains is our own). We also received valuable assistance and encouragement from Martin Hyland, Eugenio Moggi and Pino Rosolini.

## 2 Dl-domains and event structures

A poset \(\langle D, \subseteq \rangle\) having a least element \(\bot\) is said to be *complete* (and we say that \(D\) is a *cpo* if every directed subset \(M \subseteq D\) has a least upper bound \(\bigcup D\). A monotone function \(f : D \rightarrow E\)