Fixed points of Büchi automata

Mads Dam*

Department of Computer Science, University of Edinburgh, U.K.

Abstract. We give a new and direct proof of the equivalence between the linear time \( \mu \)-calculus \( \nu \mathrm{TL} \) and Büchi automata. Constructions on automata are given which compute their least and greatest fixed points. Together with other well-known constructions corresponding to the remaining \( \nu \mathrm{TL} \) connectives the result is a representation of \( \nu \mathrm{TL} \) as Büchi automata which in contrast to previously known constructions is both elementary and compositional. Applications to the problem of completely axiomatising \( \nu \mathrm{TL} \) are discussed.

1 Introduction

The relation between automata as devices for recognising behaviours, and fixed points, or equations, as means of characterising them is an important recurring theme in the theory of computation. The \( \omega \)-regular languages provides an example of particular interest in concurrency theory. They are characterised on the one hand by formulas in the linear time \( \mu \)-calculus. This logic, known as \( \nu \mathrm{TL} \), augments linear time logic by least and greatest fixed points of formally monotone contexts. The \( \omega \)-regular languages are also exactly the languages recognised by Büchi automata, finite automata applied to words of infinite length. Both \( \nu \mathrm{TL} \) and Büchi automata have had considerable attention as formalisms for specifying and verifying concurrent programs (c.f. [1, 2, 5, 9, 12, 19]).

We suggest examining the connection between \( \nu \mathrm{TL} \) and Büchi automata further. Büchi automata at present lacks a structural theory which is usable in practice, for instance for machine implementation or to support equational reasoning. The equivalence with \( \mathrm{S1S} \), the monadic second-order theory of successor, is nonelementary [10] and thus offers little concrete assistance. The linear time \( \mu \)-calculus is potentially much more valuable for this purpose. Fixed points, on the other hand, can be very troublesome in practical use. Already at the second level of alternation formulas can become highly unintelligible. Automata can prove useful aids for visualising fixed point properties.

The value of a compositional, or syntax-directed approach in such an enterprise is well documented. Indeed Büchi's original work on the decidability of \( \mathrm{S1S} \), the monadic second-order theory of one successor [3], gave a compositional representation of \( \mathrm{S1S} \) formulas as automata, representing second-order quantification, in particular, by projection. The present paper can be viewed

* Research supported by SERC grant GR/F 32219. Current address: Swedish Institute of Computer Science, Box 1263, S-164 28 Kista, Sweden. E-mail: mfd@se.sics.
as an adaptation of Büchi’s work to $\nu$TL, by providing representations for the fixed point quantifiers. That is, given an automaton recognising the language expressed by the $\nu$TL-formula $\phi$ where $\phi$ is formally monotone in the variable $X$, we produce automata recognising the least and greatest fixed points, $\mu X.\phi$ and $\nu X.\phi$ respectively, of the operator $\lambda X.\phi$. Of course only one fixed point construction, for instance for greatest fixed points, is needed due to the equivalence $\mu X.\phi \equiv \neg \nu X.\neg\phi [\neg X/X]$. However, the construction for least fixed points generalises the construction for greatest fixed points in a natural way, and by using it the need for explicit complementation of Büchi automata can be dispensed with.

Existing proofs that formulas in $\nu$TL define $\omega$-regular languages give constructions of Büchi automata that are either nonelementary because $\Sigma_1^\omega$ is used as an intermediate step, or noncompositional. The latter is the case, in particular, for the automata-theoretic techniques of e.g. [16, 18]. Their approach is global rather than compositional: The automaton for a formula $\phi$ is built as the intersection of an automaton that checks local model conditions with the complement of an automaton that checks for non-well-foundedness of a certain regeneration relation.

The paper is organised as follows: In section 2 we introduce $\nu$TL, and in section 3 we introduce Büchi automata and show how they can be represented in $\nu$TL. This representation is instructive in showing results that do not appear to be widely known, such as the collapse of the fixed point alternation hierarchy (on level $\nu \mu$), and the expressive equivalence of the aconjunctive fragment of $\nu$TL with the full language (see [7] for a definition of aconjunctivity). The fixed point constructions first builds an intermediate automaton with nonstandard acceptance conditions. This construction is described in section 4, and then in sections 5 and 6 the constructions for greatest and least fixed points are given. Finally, in section 7, we discuss the application of our construction to the problem of completely axiomatising $\nu$TL. This is of particular interest since automata-based techniques, despite their success in temporal logic in general, have not so far proved very useful where axiomatisations are concerned. The axiomatisation we have in mind is based on Kozen’s axiomatisation of the modal $\mu$-calculus [7]. Using our construction Büchi automata can be viewed as normal forms for $\nu$TL, suggesting a strategy for proving completeness whereby each formula is proved equivalent to its normal form using only the axioms and rules of inference provided. We have so far used this strategy successfully to prove completeness for the aconjunctive fragment. Our approach is related to Siefke’s completeness result for $\Sigma_1^\omega$ [14] and to Kozen’s recent completeness result for the algebra of regular events [8].

2 The Linear Time $\mu$-calculus

Formulas $\phi$, $\psi$, $\gamma$ of the linear-time $\mu$-calculus $\nu$TL are built from propositional variables $X$, $Y$, $Z$, boolean connectives $\rightarrow$ and $\land$, the nexttime operator $O$, and the least fixed point operator $\mu X.\phi$, subject to the formal monotonicity condition