Fibred Tableaux for Multi-Implication Logics

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Abstract. We investigate the notion of fibred tableaux which naturally arises from the idea of fibred semantics. Different implication operators peacefully cohabit and co-operate within the same labelled tableau method.

1 Introduction

The last decade has seen a proliferation of different logical systems proposed for a variety of different purposes, both theoretical and practical. These logics appear to satisfy the needs of different application areas or to capture different interpretations of the logical operators. This is particularly apparent in the case of the conditional operator. All the different implication logics which have been proposed in the literature seem to succeed in modelling some aspect of the ordinary use of this operator, or in suggesting useful non-standard interpretations. What emerges from these developments is a class of operators bearing a family resemblance to each other, each of which may fit a different application. So, a crucial problem, both in pure and applied logic, is that of developing a general methodology for combining logical systems—each with its own operators and its own semantics—into larger systems of "modular reasoning" where the user can easily switch from one context to the other. The proof-theoretical side of this problem can be formulated as follows: how can we combine different proof systems (e.g. tableau systems) into more general ones, possibly preserving the desirable properties of the originals?

In [DG94] we presented a method for generalizing the classical KE system [DM94] into a labelled refutation system in the spirit of Gabbay's methodology of Labelled Deductive Systems [Gab91] (but also of Fitting's prefixed tableaux [Fit83]). In that paper different "substructural" logics were characterized by different labelling algebras, within the same uniform proof-theoretical framework, but did not combine with each other to generate hybrid systems. In [Gab96, Gab95d] Gabbay investigates the general notion of fibred semantics which provides the theoretical foundations for the development of such hybrids.

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In this position paper we merge these two lines of research and investigate the notion of “fibred tableau” that arises from their interaction. In Sections 2, 3 and 4 we briefly review the KE system, our underlying classical system, and its labelled version for substructural implication logics. In Sections 5, 6 and 7 we present our new method of combining these implication logics by applying (and adapting) the principles of fibred semantics and of the labelled KE method. Finally, in Section 8 we work out a few examples of refutations in some hybrid systems.

2 The classical KE system

The system KE, like the tableau method and resolution, is a refutation system for classical logic. Unlike resolution, however, KE is not restricted to clausal form and, unlike the tableau method, it includes a cut rule which cannot, in general, be eliminated. This classical cut rule is called PB (from Principle of Bivalence) and has the following forms, depending on whether we deal with signed or unsigned formulae:

\[ T \vdash F \quad A \vdash \neg A \]

The formula A introduced by an application of this rule is called PB-formula or cut formula\(^2\). Once such a cut rule has been allowed, the branching elimination rules typical of the tableau method become unnecessarily strong and can be replaced by weaker non-branching rules (with two premises).

The rules of the KE system (for unsigned formulae) are illustrated in Table 1. The two-premiss elimination rules correspond to familiar principles of inference: modus ponens, modus tollens, disjunctive syllogism its dual. The one-premiss elimination rules are the same as the tableau rules. A KE-refutation of a set of formulae \( \Gamma \) is, as usual, a closed tree of formulae constructed according to the rules of KE starting from formulae in \( \Gamma \).

A crucial property of KE is the analytic cut property: the applications of the cut rule can be restricted to subformulae of the formulae occurring above in the branch without loss of completeness. This property allows for systematic and efficient refutation procedures. Indeed, results in [DM94] imply that any refutation procedure which can be formulated in terms of the tableau rules can be efficiently (linearly) simulated by means of the KE rules, but there are efficient and systematic KE-procedures which cannot be polynomially simulated by means of the tableau rules. One of these procedures is the canonical procedure, consisting in giving priority to the linear elimination rules over the cut rule, so that the cut rule is applied only when no elimination rule is further applicable, and the choice of the cut formulae is restricted to pairs \( A, \neg A \) such that \( A \)

\(^2\) To see that PB plays the same role as the cut rule in the classical sequent calculus, think of it as a rule that allows one to construct a closed KE-tree for \( \Gamma \), given a closed KE-tree for \( \Gamma, A \) and a closed KE-tree for \( \Gamma, \neg A \); for a discussion of this point see [DM94].