7.1 Introduction

In Section 4.7, it was shown how DPCA can be applied to develop an autoregressive with input ARX model and to monitor the process using the ARX model. The weakness of this approach is the inflexibility of the ARX model for representing linear dynamical systems. For instance, a low order autoregressive moving average ARMA (or autoregressive moving average with input ARMAX) model with relatively few estimated parameters can accurately represent a high order ARX model containing a large number of parameters [137]. For a single input single output (SISO) process, an ARMAX\((h)\) model is:

\[
y_t = \sum_{i=1}^{h} \alpha_i y_{t-i} + \sum_{i=0}^{h} \beta_i u_{t-i} + \sum_{i=1}^{h} \gamma_i e_{t-i} + e_t
\]  

(7.1)

where \(y_t\) and \(u_t\) are the output and input at time \(t\), respectively, \(\alpha_1, \ldots, \alpha_h, \beta_1, \ldots, \beta_h,\) and \(\gamma_1, \ldots, \gamma_h\) are constant coefficients, and \(e_t\) is a white noise process with zero mean [224]. For an invertible process, the ARMAX\((h)\) model can be written as an infinite order ARX model [224]:

\[
y_t = \sum_{i=1}^{\infty} \pi_i y_{t-i} + \sum_{i=0}^{\infty} \rho_i u_{t-i} + e_t.
\]  

(7.2)

The constant coefficients \(\pi_1, \pi_2, \ldots\) and \(\rho_1, \rho_2, \ldots\) are determined from the coefficients in (7.1) via the backshift and division operations [224].

The classical approach to identifying ARMAX processes requires the \textit{a priori} parameterization of the ARMAX model and the subsequent estimation of the parameters via the solution of a least squares problem [137]. To avoid over-parameterization and identifiability problems, the structure of the ARMAX model needs to be properly specified; this is especially important for multivariable systems with a large number of inputs and outputs. This structure specification for ARMAX models is analogous to specifying the observability (or controllability) indices and the state order for state space models, and is not trivial for higher order multivariable systems [215]. Another problem with the classical approach is that the least squares problem
Canonical Variate Analysis requires the solution of a nonlinear optimization problem. The solution of the nonlinear optimization problem is iterative, can suffer from convergence problems, can be overly sensitive to small data fluctuations, and the required amount of computation to solve the optimization problem cannot be bounded [131].

To avoid the problems of the classical approach, a class of system identification methods for generating state space models called subspace algorithms has been developed in the past few years. The class of state space models is equivalent to the class of ARMAX models [8, 137]. That is, given a state space model, an ARMAX model with an identical input-output mapping can be determined, and vice versa. The subspace algorithms avoid a priori parameterization of the state space model by determining the states of the system directly from the data, and the states along with the input-output data allow the state space and covariance matrices to be solved directly via linear least squares [215] (see Figure 7.1). These algorithms rely mostly on the singular value decomposition (SVD) for the computations, and therefore do not suffer from the numerical difficulties associated with the classical approach.

![Diagram](image-url)

**Fig. 7.1.** A comparison of the subspace algorithm approach to the classical approach for identifying the state space model and extracting the Kalman states [216]