Quantum MV Algebras*

Abstract. We introduce the notion of quantum MV algebra (QMV algebra) as a generalization of MV algebras and we show that the class of all effects of any Hilbert space gives rise to an example of such a structure. We investigate some properties of QMV algebras and we prove that QMV algebras represent non-idempotent extensions of orthomodular lattices.

Key words: MV algebra, Multi-valued logic, quantum MV algebra, unsharp orthoalgebra, unsharp quantum mechanics

Introduction

MV algebras (multi valued algebras) were introduced by Chang ([3]) in order to provide an algebraic proof of the completeness theorem of infinite-valued logic $L_\infty$ (Lukasiewicz logic). The “privileged” model of $L_\infty$ is the real interval $[0,1]$; the numbers of this interval are interpreted, after Lukasiewicz, as the possible truth values we can assign to the logical sentences. Accordingly, infinite-valued logic represents a strong generalization of bivalent (classical) logic.

Birkhoff and von Neumann ([1]) used the $\{0,1\}$-analogy to justify the interpretation of projections of a Hilbert space as the “experimental propositions” pertaining to a quantum physical system. The basic reason underlying this interpretation is that projections are idempotent, self-adjoint and bounded linear operators whose spectrum is contained in $\{0,1\}$. In standard quantum mechanics, physical quantities are mathematically described by self-adjoint operators. The spectrum of a self-adjoint operator, representing a physical quantity, is interpreted as the set of all possible values that the physical quantity may assume in a given state. Accordingly, it is quite natural to consider projections as “experimental propositions”, i.e. $\{0,1\}$-physical quantities.

The question arises whether there is any “quantum counterpart” of the real interval $[0,1]$ that could be interpreted as a generalization of the sharp

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(i.e. \{0, 1\}) experimental propositions. The natural framework for such a generalization is represented by the so called unsharp or operational approach to quantum mechanics ([8], [20]). One of the basic ideas of this approach is the “liberalization” of the mathematical counterpart for the intuitive notion of “experimentally testable proposition”. As we have seen, in orthodox Hilbert-space quantum mechanics, experimental propositions are mathematically represented as projections \( P \) in the Hilbert space \( \mathcal{H} \), corresponding to the physical system \( S \) under investigation. Let \( P \) be a projection representing a proposition and let \( D \) be a density operator representing a state of \( S \). The number \( \text{Tr}(DP) \) can be interpreted as the probability-value that the system \( S \) in the state \( D \) verifies \( P \) (Born probability).

However, projections are not the only operators for which a “Born probability” can be defined. Let us consider the class \( E(\mathcal{H}) \) of all linear bounded operators \( E \) s.t.

\[
\text{Tr}(DE) \in [0,1],
\]

for any density operator \( D \).

It turns out that \( E(\mathcal{H}) \) properly includes the set \( P(\mathcal{H}) \) of all projections of \( \mathcal{H} \). In a sense, \( E(\mathcal{H}) \) represents a “maximal” possible notion of experimental proposition, in agreement with the probabilistic rules of quantum theory. In the framework of the unsharp approach, the elements of \( E(\mathcal{H}) \) have been called effects.

In the last years, some algebraic structures have been proposed to model the class of all effects. Giuntini and Greuling ([15]) developed a partial structure, called weak orthoalgebra, as an attempt to construct a formal language for unsharp properties. Such a structure (called also unsharp orthoalgebra or effect algebra) has been further investigated in [7] and [9]. Recently, Kopka and Chovanec ([19]) have proposed a new candidate (called \( D\)-poset) to represent algebraically the class of effects. It turns out that the category of \( D\)-posets is isomorphic to the category of unsharp orthoalgebras. (see [7]). Another structure (called Brouwer-Zadeh poset) has been proposed by Cattaneo and Nisticò ([3]) and further investigated in [12] and [13]. The main feature of Brouwer-Zadeh posets is represented by the splitting of the complement into two kinds of non-standard orthocomplements. These complements collapse into the same operation whenever they are restricted to the class of all sharp properties (projections) of a Hilbert space.

In this paper we will propose a class of algebraic structures (called quantum MV algebras) that generalize MV algebras; the class of all effects of any Hilbert space turns out to be an instance of these structures. Quantum MV algebras (QMV algebras) retain some important properties of MV algebras, while violating the crucial axiom of MV algebras: the so called