Conditional Theories

Abstract. This paper introduces conditional logic, that is, a variant of free logic where the existence condition of a function is defined by a formula of the formal language. Syntax and semantics are developed. A completeness theorem is given.

0. Introduction

In classical as well as intuitionistic logic there are relatively few publications concerning partially defined elements. One of the reasons is that in classical logic there is no need to introduce formal languages whose models are partial algebras, as every partial functional symbol can be conveniently extended to a total one.

However, in intuitionistic logic, the usefulness of an appropriate language for partial elements has been shown by Dana Scott [77] where an existence predicate $E$ is introduced. Next, $ETT$, the elementary theory of topoi, which is classical as it includes the axiom of the excluded middle, gives another example\(^1\), where a partial symbol cannot be extended to a total one without great inconvenience. As shown by Blanc [80], Freyd [76], an appropriate formal language for categories does not contain an equality of objects, (suppose $ETT$ expressed in an appropriate formal language Blanc [80], Preller [75]).

Now, let $\Pi(f, g)$ be the pullback of $f : Y \to A$ and $g : Z \to A$. Consider the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow & & \downarrow \\
1 & \xrightarrow{\Pi(f, g)} & Z \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & Y \\
\end{array}
\]

If we introduced a functional symbol $P(u, v, f, g) : X \to \Pi(f, g)$, we could show that $0 \simeq 1$ as $P(Id 1, Id 1, T : 1 \to \Omega, \bot : 1 \to \Omega)$ is an arrow from 1

\(^1\) Communicated to us by A. Preller
to an object isomorphic to 0. So $P$ is only partial, it exists only on condition that $uf = vg$.

Formal category theory makes use of interpretations. For example slicing, the topos of objects over $A$, is such an interpretation of $ETT$ into $ETT$. There we cannot avoid partial functional symbols. However, they are all conditional in the following sense: the domain of existence is defined by a formula of the language, expressing the condition of existence.

Moreover, in Computer Sciences, partial algebras play a great role as abstract data type, program-specifications and initial algebra semantics of programming language (see Kaphengst-Reichel [77]). So, it would be convenient and even necessary in order to handle a first order language to have an adequate concept of equations, formulae and a correct notion of validity. A first order language for partial algebra has been studied by P. Burmeister [82]; he is only interested in the notion of equations. The language we propose here is more general allowing arbitrary formulae as conditions.

Sections 1, 2, 3 develop the syntax of conditional theories, and Section 4 its semantic aspects including a completeness theorem. In Section 5, we study in what manner conditional theories and Dana Scott's theories are equi-interpretable.

1. Conditional languages

1.1. Definition.

(i) A conditional first order language $L$ is given by a list of predicate symbols $V, P_1, P_2, \ldots$, $V$ being of empty degree, and a list of functional symbols $F_1, F_2, \ldots$ and the usual logical constants $\land, \neg, \exists, \equiv$. Terms and formulae are defined as usual. Then a list of formulae $\Phi_1, \Phi_2, \ldots$ is given such that $\Phi_n$ has no occurrences of functional symbols $F_i$ with $i \geq n$ and has at most $p_n$ free variables where $p_n$ is the degree of $F_n$. $\Phi_n$ is called the condition of $F_n$. The condition of $F_1$ is $V$.

(ii) We call an expression of $L$ any term or formula of $L$. Then we define the construction of an expression $\theta$ of $L$ inductively as a finite tree with a smallest element, its root.

- $V$ is the construction of $V$.
- $x$ is the construction of $x$ for any variable $x$.
- If $T_1, \ldots, T_n$ are the constructions of the terms $t_1, \ldots, t_n$ respectively, and if $P$ is a predicate symbol of degree $m$, then the tree $T$ of root $Pt_1 \ldots t_m$ and of immediate subtrees $T_1, \ldots, T_m$ is the construction of $Pt_1 \ldots t_m$.
- If $T_1$ and $T_2$ are the constructions of the terms $t_1$ and $t_2$, then the tree $T$

\[ \text{Notation: } \land, \neg, \text{ and } \forall \text{ are defined as usual.} \]