Formal Asymptotic Solution of a Singularly Perturbed Nonlinear Optimal Control Problem

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Abstract. A system of equations that arises in a singularly perturbed optimal control problem is studied. We give conditions under which a formal asymptotic solution exists. This formal asymptotic solution consists of an outer expansion and left and right boundary-layer expansions. A feature of our procedure is that we do not a priori eliminate the control function from the problem. In particular, we construct a formal asymptotic expansion for the control directly. We apply our procedure to a Mayer-type problem. The paper concludes with a worked example.

Key Words. Optimal control, singular perturbations, boundary-layer techniques, two-point boundary-value problem.

1. Introduction

In this paper, we study a system of differential equations which arises in the application of the Pontryagin principle (Ref. 1) to a nonlinear control problem involving a small parameter. We let \( H(t, x, \lambda, y, \mu, u, \epsilon) \) be a scalar-valued \( C^\infty \) function, and we consider the system of equations

\[
\begin{align*}
\dot{x}(t, \epsilon) &= f(t, x, y, u, \epsilon), \\
\dot{\lambda}(t, \epsilon) &= -H_x(t, x, \lambda, y, \mu, u, \epsilon), \\
\epsilon \dot{y}(t, \epsilon) &= g(t, x, y, u, \epsilon), \\
\epsilon \dot{\mu}(t, \epsilon) &= -H_y(t, x, \lambda, y, \mu, u, \epsilon),
\end{align*}
\]

where \( 0 \leq t \leq T, \epsilon \) is a small positive parameter, \( x \in E^{n_1}, \lambda \in E^{n_1}, y \in E^{n_2}, \mu \in E^{n_2}, \) and \( u \in E^{n_3} \). The functions \( f \) and \( g \) are \( C^\infty \). The functions \( x, \lambda, y, \) and

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The function $u(t, \epsilon)$ appearing in (1) represents the optimal control choice for some underlying control problem. According to the Pontryagin principle, it minimizes the scalar-valued function $H$, called the Hamiltonian. The control $u(t, \epsilon)$ takes values in a bounded domain $D \subset E^n$ with compact boundary.

Singular perturbation techniques have been widely used to study systems of differential equations containing a small parameter. A general discussion of these techniques can be found in Wasow (Ref. 2), Cole (Ref. 3), and Nayfeh (Ref. 4). A boundary-value problem for a system of differential equations with small parameter appearing in the derivative has been treated by Tupciev (Ref. 5). More closely related to the problem considered in this paper are the work of Hoppensteadt (Ref. 6), O'Malley (Ref. 7), Hadlock (Ref. 8), Sannuti (Ref. 9), and Harris (Ref. 10). In the problems considered by these authors, the control $u(t, \epsilon)$ can either be found explicitly in terms of $x, \lambda, y,$ and $\mu$ or can be found in principle in terms of the remaining functions. Once $u(t, \epsilon)$ is replaced by a known function of $x, \lambda, y,$ and $\mu$ in the equations, they commence their asymptotic analysis of the problem. In contrast to this, we do not a priori eliminate the control $u$ from the problem. We construct a formal asymptotic expansion for $u(t, \epsilon)$ in a direct manner.

In Sections 2 and 3, we give a detailed description of the problem and define precisely what we mean by outer expansion and boundary-layer expansions. In Sections 4 and 5, we describe a recursive procedure for the construction of a formal asymptotic series solution of (1) which satisfies certain boundary conditions and an extremum condition. This asymptotic series solution consists of an outer expansion and a left and right boundary-layer expansion. In Section 6, we discuss the matching of the outer expansion and the boundary-layer expansions. We describe, in Section 7, an application of our procedure to a Mayer problem (Ref. 1) in which various assumptions that we make are satisfied. Finally, in Section 8, we carry out the details of the construction in order to obtain the leading terms of an asymptotic expansion for an explicit problem.

The asymptotic validity of the formal expansion derived in this paper is proved by Freedman and Kaplan (Ref. 11).

2. Full Problem and Reduced Problem

We let $D \subset E^n$ be a closed bounded domain. We define

$$\hat{D} = \{v(t) \mid v(t) \in D \forall t \in [0, T]\}.$$