A Necessary and Sufficient Condition for
Pareto-Optimal Security Strategies in
Multicriteria Matrix Games\textsuperscript{1,2}

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Abstract. In this paper, a scalar game is derived from a zero-sum multicriteria matrix game, and it is proved that the solution of the new game with strictly positive scalarization is a necessary and sufficient condition for a strategy to be a Pareto-optimal security strategy (POSS) for one of the players in the original game. This is done by proving that a certain set, which is the extension of the set of security level vectors in the criterion function space, is convex and polyhedral. It is also established that only a finite number of scalarizations are necessary to obtain all the POSS for a player. An example is included to illustrate the main steps in the proof.

Key Words. Game theory, multicriteria games, games with vector payoffs, Pareto-optimal security strategies, multicriteria optimization, scalarization methods.

1. Introduction

A natural extension of the well-known classical zero-sum matrix game (with a scalar criteria), proposed and solved by von Neumann, is the zero-sum matrix game with a vector payoff, which is also known as the zero-sum multicriteria matrix game. In Blackwell's paper (Ref. 1), an asymptotic analog of the minimax theorem in scalar criterion games was established for repeated games with vector payoffs. The analysis was aimed

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at answering the question as to whether a player will be able to force his average payoff to approach or exclude a given subset in the payoff space, if the game is repeated a large number of times. Shapley (Ref. 2) defined the concept of equilibrium points in these games and presented methods of obtaining them through the solution of scalarized bimatrix games. A similar approach was taken by Nieuwenhuis (Ref. 3) and Corley (Ref. 4) using the notion of Pareto optimality (efficiency). The equilibrium points, as obtained in Refs. 2–4, do not possess the important property of security in the individual criteria against opponent's deviations in strategy, unlike the equilibrium saddle points in zero-sum scalar criterion matrix games. In Ref. 5, a solution concept based on Pareto optimality and security was proposed which is independent of the notion of equilibrium. It was demonstrated that this concept is important in some areas of application. A similar concept has been used earlier by Schmitendorf and Moriarty (Ref. 6) for coalitive Pareto optimality and by Schmitendorf (Ref. 7) to analyze systems with disturbances. But these were in the context of differential games.

In Ref. 5, Pareto-optimal security strategies (POSS) were obtained for a player in a zero-sum multicriteria matrix game by scalarization of the original game. A necessary condition and a sufficient condition were separately obtained. In this paper, we prove that strictly positive scalarization is both a necessary and a sufficient condition for such games. In addition, we also prove that only a finite number of scalarizations are required to obtain all the Pareto-optimal security strategies of a player.

The paper is organized as follows. Section 2 formulates the multicriteria game and defines Pareto-optimal security strategies. Section 3 defines a new scalarized game and a set which is an extension of the set of security level vectors associated with the multicriteria game. It is established that, if the extended set is polyhedral, then the solution of the new game with strictly positive scalarization is a necessary and sufficient condition for a strategy to be a POSS. In Section 4, we prove that the extended set is always polyhedral for a multicriteria matrix game. Section 5 presents an example illustrating the major steps in the proof, and Section 6 concludes the paper.

2. Some Definitions and Remarks

Let \( \mathbb{D} \subseteq \mathbb{R}^n \) be a compact, convex subset of the \( n \)-dimensional real space. An element \( v \in \mathbb{D} \) is called a vector and is an \( n \)-tuple \( (v_1, \ldots, v_n) \) of real numbers. Let \( y = (y_1, \ldots, y_n) \) and \( z = (z_1, \ldots, z_n) \) be two arbitrary vectors in \( \mathbb{D} \). Then, using the definitions given in Lin (Ref. 8), but with a slight change in notation, we have:

\[
\begin{align*}
(i) \quad y \succeq z, & \iff y_i \geq z_i, \text{ for all } i; \\
\end{align*}
\]