Abstract. We present tableau systems and sequent calculi for the intuitionistic analogues $IK$, $ID$, $IT$, $IKB$, $IKDB$, $IB$, $IK4$, $IKD4$, $IS4$, $IKB4$, $IK5$, $IKD5$, $IK45$, $IKD45$ and $IS5$ of the normal classical modal logics. We provide soundness and completeness theorems with respect to the models of intuitionistic logic enriched by a modal accessibility relation, as proposed by G. Fischer Servi. We then show the disjunction property for $IK$, $ID$, $IT$, $IKB$, $IKDB$, $IB$, $IK4$, $IKD4$, $IS4$, $IKB4$, $IK5$, $IK45$ and $IS5$. We also investigate the relationship of these logics with some other intuitionistic modal logics proposed in the literature.

1. Introduction

In this paper we provide tableau systems and sequent calculi for normal propositional intuitionistic modal logics. The semantics is defined through Kripke-type models, where the forcing relation on modal connectives and the frame properties are those proposed by Fischer Servi in [13]. The frames, endowed with two accessibility relations, are those of intuitionistic logic enriched by a Kripkean accessibility relation. Given any combination of the properties of seriality, reflexivity, transitivity, symmetry and euclideanness for the modal accessibility relation, an analogue of the classical modal system is defined.

We use the notations and the method due to Fitting [14]. Strong completeness theorems are provided with the assumption of the cut rule. The cut is not eliminable for the systems with symmetry and for both $IK5$ and $IKD5$.

Although the literature on intuitionistic modal logics does not seem to be very extensive, there are several proposals which define intuitionistic analogues of classical modal logics. Here we briefly survey the literature that is strictly related to the present work.

The problem of presenting an intuitionistic concept of modality was faced by Fitch and Prior in early papers [15, 24]. Prior proposes a modal extension

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of the intuitionistic propositional calculus which turns out to be S5 once the excluded middle is added. In [2, 3] Bull studies Prior's propositional modal calculus and shows that it is related to the intuitionistic monadic calculus just as S5 is related to the classic monadic calculus. The relationship between intuitionistic modal logics and intermediate predicate logics has been further investigated for instance in [22, 25, 26].

In [21] a lattice of intuitionistic modal logics is given, with the necessity operator □ as primitive, both in Hilbert and in Gentzen-style formulations. These logics are intermediate between the intuitionistic S4 and S5 in the sense of [3]. It is worth noticing that in [21] a semantics with a Kripke-type frame is also adopted. The modal relation is introduced as an extension of the intuitionistic accessibility relation.

General criteria to find the intuitionistic counterpart of many classical modal systems are both proposed in [1, 6] and in [8, 9, 10, 11, 12].

In [1] two systems HK□ and HK◊ are introduced, as intuitionistic counterparts of the classical system K. The first one deals only with □ and the latter only with ◊. HK◊ turns out to be sound and complete with respect to the Kripke-type frames satisfying one of the two connecting properties we are concerned with (see the second condition of Definition 1 and [13]). On the other hand, HK□ is sound and complete with respect to three classes of frames ordered by inclusion and where the forcing relation for □ is given as in classical logic. When bringing together both the property of HK◊ and the strongest connecting property of HK□-frames (i.e. the strictly condensed frames or also □-normal logics in [28]), ◊ turns out to be definable with □ as primitive. Here, it is further shown that this logic, called HK□◊, is stronger than the logic obtained from HK□ by adding the axiom scheme ◊ ∧ □¬□¬ (see also [16]). We show that HK□◊ can also be obtained by a proper subclass of IK-frames of [13] (See Remark 2). Further studies on these logics can be found in [6, 16, 23].

In Fisher Servi's works the two modal operators of possibility ◊ and necessity □ are required to be non–interdefinable. Two connecting axioms are proposed to strengthen the weak connection between them. As one should expect from the intuitionistic behaviour of quantifiers, both □¬¬A ⊃ ◊A and ¬◊¬A ⊃ □A, as well as □¬A ⊃ ◊¬A, cannot be derived on an intuitionistic basis. Following such a proposal, in [13] some intuitionistic modal logics and Kripke-type semantics are given. It is further shown that IS5 is equivalent to Bull's MIPC (see [2, 3, 21]). The forcing relations for both modal operators are analogous to those of the intuitionistic quantifiers on Kripke frames ([5], see Definition 3.6). Two connecting properties for the Kripke-type frames are given to characterize the intuitionistic modal logics.