Variational bounds on energy dissipation in an incompressible channel flow with uniform injection and suction

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Summary. A variational principle for lower bounds on the time-averaged mass flux in an incompressible channel flow with uniform wall injection and suction is derived from the incompressible Navier-Stokes equations. With appropriate test background flow fields, the explicit estimates for the friction coefficient are produced and discussed.

1 Introduction

While it is not feasible to obtain exact solutions to the equations of motion for turbulent flows, it is possible to derive mathematically rigorous upper bounds on certain quantities characterizing turbulent flow fields, such as the rate of energy dissipation in shear flows or the rate of heat transfer in turbulent convection. A detailed theory of upper bounds has been developed by Howard [1], and Busse [2]. For instance, this theory still gives for asymptotically high Reynolds numbers the best upper bound on the rate of energy dissipation in turbulent Couette flow that has been calculated so far. Recently the other approach by Doering and Constantin for bounding the energy dissipation rate in turbulent shear flow [3], channel flow [4], and convection [5] has sparked renewed interest in the theory of upper bounds. Kerswell [6] has elucidated a formal mathematical connection between these approaches.

Fluid problems involving sources and sinks appear in many disciplines and engineering applications. Injection and suction are used as devices to control boundary layer development. For example, injection from the permeable wall has been found to be an effective tool to produce film cooling for turbine blades exposed to a hot free stream. Because of the fluid injection into the mainstream, a thickened boundary layer is created, and consequently, the surface skin friction and hence the drag decreases. An elevated level of turbulent kinetic energy is also observed. In aeronautical applications, suction, on the other hand, is frequently used to delay separation and to inhibit the transition to turbulence. Even though the magnitude of the transpiration rate is often low compared with the mainstream, it significantly changes the surface skin friction as well as the turbulence quantities near the wall. In [7], the rate of viscous energy dissipation in a shear layer with injection and suction was studied by means of exact solutions, nonlinear and linearized stability theory, and rigorous upper bounds by Doering et al. The objective of the present paper is to use the variational principle suggested by Doering and Constantin to derive a lower bound on the time-averaged mass flux in channel driven by constant pressure gradient and uniform wall injection and suction and to produce explicit estimates for the friction coefficient.
2 Variational principle

To investigate the influence of injection and suction on channel flows, we suppose an incompressible Newtonian fluid is confined to a rectangular channel. A uniform pressure gradient of magnitude $P/L_x$ in the $x$-direction drives the flow. Without loss of generality mass units can be chosen so that the density $\rho = 1$, and we denote the kinematic viscosity by $\nu$. The fluid’s velocity vector field $u(x, t) = (u_x, u_y, u_z)$ satisfies the Navier-Stokes equations

\begin{equation}
\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \nu \Delta u + i \frac{P}{L_x},
\end{equation}

\begin{equation}
\nabla \cdot u = 0,
\end{equation}

where $p(x, t)$ is the pressure field determined by the divergence-free condition on $u$, and $\mathbf{i}$ is the unit vector in the $x$-direction. There are uniform injection and suction of fluid into the layer with speed $V$ on the top and bottom planes. The conditions at the rigid boundaries are thus

\begin{equation}
u = -jV \quad \text{at} \quad y = 0 \quad \text{and} \quad y = h.
\end{equation}

In this work, we restrict our attention to periodic boundary conditions on all dependent variables in the horizontal directions with periods $L_x$ and $L_z$. In addition, we define a control parameter – the Reynolds number $Re_V = \frac{hV}{\nu}$.

The instantaneous mass flux in the $x$-direction is

\begin{equation}
\Phi(t) = \int_0^{h} dy \int_0^{L_z} dz \cdot u(x, y, z, t).
\end{equation}

In light of the incompressibility condition and the boundary conditions, this flux is independent of $x$. The definitions of $L_2$ norm and the long time average of a function $f(x, y, z, t)$ are same as that in [4], [7].

The evolution equation for the kinetic energy in the fluid is derived from the Navier-Stokes equation (2.1) by dotting with $u$, integrating over space, and integrating by parts using the boundary conditions:

\begin{equation}
\frac{d}{dt} \frac{1}{2} ||u||^2 = P\phi + V \int_0^{L_x} dx \int_0^{L_z} dz |p|_{y=h} - |p|_{y=0} - \nu ||\nabla u||^2.
\end{equation}

The terms on the right-hand side of this equation are, in order, the power expended by the applied pressure driving the fluid flow, the rate of work performed by the injection and suction processes, the power removed by viscous dissipation in the fluid.

The approach by Doering and Constantin [3]-[5], [7] rests on a decomposition of the velocity field $u(x, t)$ into a steady background flow and a time dependent fluctuations,

\begin{equation} u(x, t) = iU(y) - jV + \nu(x, t), \end{equation}

where $\nu$ satisfies the same boundary conditions as $u$ and $U$ vanishes at $y = 0$ and $h$. We introduce the decomposition (2.6) into (2.1) and (2.2) to obtain

\begin{equation}
u + \nu \cdot \nabla u + iu_y U'(y) + [U(y) \partial_x - V \partial_y] v - iV U'(y) + \nabla p = \nu \Delta u + iu U''(y) + i \frac{P}{L_x},
\end{equation}

\begin{equation}\nabla \cdot u = 0.
\end{equation}