Inverse Matroid Intersection Problem

CAI MAO-CHENG AND YANJUN LI
Institute of Systems Science, Academia Sinica, Beijing 100080, China

Abstract: Let $M_1$ and $M_2$ be matroids on $S$, $B$ be their $k$-element common independent set, and $w$ a weight function on $S$. Given two functions $b > 0$ and $c > 0$ on $S$, the Inverse Matroid Intersection Problem (IMIP) is to determine a modified weight function $w'$ such that (a) $B$ becomes a maximum weight common independent set of cardinality $k$ under $w'$, (b) $c|w' - w|$ is minimum, and (c) $|w' - w| \leq b$. Many Inverse Combinatorial Optimization Problems can be considered as the special cases of the IMIP.

In this paper we show that the IMIP can be solved in strongly polynomial time, and give a necessary and sufficient condition for the feasibility of the IMIP. Finally we extend the discussion to the version of the IMIP with Multiple Common Independent Sets.

Key Words: Inverse matroid intersection problem, minimum cost circulation, strongly polynomial algorithm.

The notations and terminologies used in this paper are the same as in [4] and [7].

The Inverse Combinatorial Optimization Problems (ICOP) have attracted more attentions of researchers recently, because the research on the ICOP has not only important value on theory but much motivation from practice. For example, in a transportation network, some roads are very crowded, but some roads are relatively uncrowded. To overcome the traffic jams: one way is to build new roads; another is to change the transportation cost. For example, we can increase the cost on overcrowded road, or decrease the cost on uncrowded road to divide traffic flow, and reduce the traffic jams. Obviously, the first method needs large investments, while the second is also feasible under certain conditions. Of course, we hope the change in cost won’t be too large under the condition that we solve the traffic jams.

The ICOP was first proposed by D. Burton and Ph. L. Toint in [1], after that J. Zhang, Z. Ma, Z. Liu and the others have done some research works on the inverse problems of shortest path, bipartite matching, minimum spanning tree, minimum arborescence and etc (see [2], [5], [6], [8], [9], [14] and [15]). In this paper we will consider a more general model of the ICOP: the Inverse Matroid

1 Research partially supported by the National Natural Science Foundation of China
Intersection Problem (IMIP). It is well-known that the matroid intersection is a powerful generalization of many combinatorial optimization problems. For instance, the shortest path problem can been reduced to a bipartite matching problem (see [7, pp. 186–187]). The minimum spanning tree problem is a special case of a graphic matroid (see [7, p. 266] or [11, pp. 282, 285]). And the bipartite matching and the minimum arborescence problems are special cases of matroid intersections (see [7, pp. 301, 304] or [11, pp. 289–290]). So all the above inverse problems are just special cases of the IMIP. It will be shown that the IMIP can be formulated as a combinatorial linear program and also be transformed into a minimum cost circulation problem. Hence it can be solved by a combinatorial strongly polynomial algorithm. We also give a necessary and sufficient condition for the feasibility of the IMIP. Finally we extend the discussion to the version of the IMIP with Multiple Common Independent Sets.

Let us fix some notations. In the following it will be always supposed that two matroids \( M_1 = (S, \mathcal{J}_1) \) and \( M_2 = (S, \mathcal{J}_2) \), their common independent set \( B \) of cardinality \( k \), a weight function \( w: S \to \mathbb{R} \), and two functions \( b, c: S \to \mathbb{R}^+ \) are given, and \( k \) fixed.

For a family \( \mathcal{F} \) of subsets of \( S \) and a function \( f \) on \( S \), \( F \in \mathcal{F} \) is said to be \( f \)-maximal in \( \mathcal{F} \) if \( \sum (f(x): x \in F) \geq \sum \{f(x): x \in X\} \) for all \( X \in \mathcal{F} \).

The IMIP can be stated as follows: Given \( M_1, M_2, B, w, b \) and \( c \) as above, we need to determine a modified weight function \( w' \) such that

(a) \( B \) becomes a \( w' \)-maximal common independent set of cardinality \( k \),
(b) \( c|w' - w| \) is minimum, and
(c) \( |w' - w| \leq b \).

Now from (a) and (b) one easily deduces that

\[
 w'(e) \geq w(e) \quad \forall e \in B \quad \text{and} \quad w'(e) \leq w(e) \quad \forall e \notin B .
\]

So we have

\[
 w'(e) = \begin{cases} 
 w(e) + \delta(e) & \text{if } e \in B , \\
 w(e) - \delta(e) & \text{otherwise ,}
\end{cases}
\]

(1)

\[
 0 \leq \delta(e) \leq b(e) \quad \forall e \in S .
\]

(2)

Our first aim is to show that the IMIP can be formulated as a combinatorial linear program and hence be solved efficiently by a strongly polynomial algorithm. We need two simple lemmas due to A. Frank.

**Lemma 1** [4]: For a given matroid \( M = (S, \mathcal{I}) \), let \( \mathcal{I}^k = \{X: X \in \mathcal{I}, |X| = k\} \). \( I \in \mathcal{I}^k \) is \( w \)-maximal in \( \mathcal{I}^k \) if and only if