A Frequency Response Function for Linear,
Time-Varying Systems*

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Abstract. It is well known that the steady-state response of a linear, time-
invariant, finite-dimensional, exponentially stable system to a periodic input sig-
nal results, after a phase shift, in a periodic output signal of the same period with
amplitude equal to the rescaling of the input amplitude by the modulus of the
value of the transfer function at the given frequency. Moreover, the phase shift of
the output signal is equal to the phase of the value of the transfer function at the
given frequency. For this reason the transfer function is also referred to as the
"frequency response function." We present an analogue of this idea for linear,
finit-dimensional, time-varying systems, in both the continuous- and discrete-
time settings. The problem of constructing a time-varying system which associates
a given output signal to each complex exponential input signal in a prescribed set
can be posed as a modeling question. This leads to a new modeling interpretation
for some of the time-varying interpolation problems which have recently been
studied in the literature and a new motivation for the study of point evaluation for
triangular operators recently introduced by Alpay, Dewilde, and Dym and by the
authors for the continuous-time case.

Key words. Time-varying systems, Input–output maps, Point evaluation, Trans-
mission zero, Transient term, Stability, Frequency response function.

1. Introduction

We consider first a linear, time-invariant, continuous-time, finite-dimensional sys-
tem given by state-space equations

\[ \dot{x} = Ax + Bu, \quad x(0) = x_0, \]
\[ y = Cx + Du. \]

When a vector periodic input signal of the form \( u(t) = u_0 e^{j\omega t} \) (where \( j = \sqrt{-1} \)) is
fed into the system, a straightforward computation reveals the output $y(t)$ to be

$$y(t) = Ce^{tA} [x_0 - (j\omega_0 - A)^{-1}Bu_0] + W(j\omega_0)u_0 e^{j\omega_0 t},$$  

(1.1)

where $W(\lambda) = D + C(\lambda - A)^{-1}B$ is the transfer function or frequency response function of the system. (Here we often identify a number $j\omega_0$ with the scalar multiple of the identity matrix $j\omega_0 I$; the meaning will be clear from the context.) In particular, if $A$ is a stable matrix (i.e., all eigenvalues of $A$ are in the open left half-plane), then the first term on the right-hand side of (1.1) decays to zero exponentially, and asymptotically $y(t)$ is again periodic with the same period $\omega_0$ and with (complex) direction vector $W(j\omega_0)u_0$ equal to the input direction vector $u_0$ multiplied by the value of the transfer function $W(j\omega_0)u_0$ at the point $j\omega_0$ on the imaginary axis corresponding to the frequency $\omega_0$. This explains the role of the transfer function as a "frequency response function." Alternatively, it may be said that there is a choice of initial condition $x_0 = (j\omega_0 - A)^{-1}Bu_0$ so that the output signal $y(t)$ is again periodic with the same period as the input signal, and with the direction vector equal to the direction vector of the input signal multiplied by the value of the transfer function of the system at the given frequency. More generally complex exponential vector input signals ($u_0 e^{\lambda t}$ with $\lambda$ not necessarily purely imaginary) may be considered and much the same analysis applied. If $A$ is stable and $\lambda_0$ is in the closed right half-plane, then the first term in (1.1) decays exponentially to zero and the second term represents the nontrivial steady-state behavior. Even if $A$ is not stable there is a choice of initial condition so that the output induced by the complex exponential input $u_0 e^{\lambda t}$ is equal to a complex exponential output $W(\lambda_0)u_0 e^{\lambda t}$ with the same complex exponent $\lambda_0$ and with the direction vector $W(\lambda_0)u_0$ obtained by multiplication of the input direction vector $u_0$ by the value $W(\lambda_0)$ of the transfer function at $\lambda_0$. Moreover $\lambda_0$ can be viewed as a complex frequency variable. For further elaboration of this point of view, we refer to [MK].

The purpose of this paper is to develop these ideas for time-varying, finite-dimensional linear systems. It turns out that the natural tool for this analysis is the notion of point evaluation for input-output operators of linear, time-varying systems introduced by Alpay et al. [ADD] (see also [BGK2]). We first present the analogue for continuous-time systems; this uses the notion of point evaluation for continuous, time-varying systems introduced by the authors (see [BGK1]).

Conversely, there is a natural inverse problem. Consider as given a finite set of "time-varying complex frequencies" together with associated input and output direction vectors. A natural open problem then is to construct a time-varying system consistent with these input–output data. An added constraint is to find a model of minimal complexity (as measured by the dimension of the state space in a state-space realization of the system) which fits the data. This issue leads to a time-varying analogue of Cauchy interpolation problems for rational matrix functions (Lagrange–Sylvester interpolation with a minimal degree constraint). The corresponding modeling problem in turn has been studied in detail by Antoulas and Willems [AW] for the time-invariant case. The associated mathematical interpolation problem has been studied in depth by Antoulas et al. [ABKW] and by Boros et al. [BSK]. Our results extend the modeling interpretation of the problem to the time-varying case. If instead a model with minimal maximum energy of