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Chern Numbers, Quaternions, and Berry's Phases in Fermi Systems

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Abstract. Yes, but some parts are reasonably concrete.

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1. Introduction

Quantum Hamiltonians that depend on parameters give rise to interesting geometric and topological questions [1–3]. A basic paradigm, due to M. Berry [1], is a spin J in a magnetic field $\mathbf{B}$:

$$H(\mathbf{B})=\mathbf{B}\cdot\mathbf{J}. \tag{1.1}$$

The geometric objects of concern are the bundles of one dimensional eigenspaces of $H(\mathbf{B})$ [2]. They are naturally defined over $\mathbf{B}\in\mathbf{R}^3/\{0\}$, with $\mathbf{B}=0$, the point of level crossings, removed. The adiabatic evolution can be used to define a natural connection for the bundles.

$H(\mathbf{B})$, being odd under time reversal, is the paradigm for the general case. It is natural to ask if time-reversal invariant Hamiltonians also give rise to interesting geometry. The answer is sensitive to statistics in the sense that bosons (i.e., integer spin systems) are different from fermions (i.e., half-odd-integer spin systems).

Mead [4] proposed the study of time-reversal invariant fermi systems, for which spin J in a quadrupole electric field is the basic paradigm [5],

$$H(Q)=Q_{\mu,\nu}J_\mu J_\nu. \tag{1.2}$$

$Q_{\mu,\nu}$ are the components of a real, 3 by 3, symmetric and traceless matrix, and we use the summation convention. In this paper, we compute the Chern numbers, curvatures, and holonomies for the bundles associated to spectral subspaces of $\{H(Q)\}$.

1.1 The Adiabatic Connection. Let $H(x)$ be

$$H(x)=x_\alpha T_\alpha, \tag{1.3}$$

where $x=(x_1,\dots,x_n)\in\mathbf{R}^n/\{0\}$ and the $\{T_\alpha\}$ are fixed self-adjoint operators.

Fix an eigenvalue $\lambda(x)$ of $H(x)$ and let $P(x)$ be the associated spectral projection:

$$P(x)=\frac{1}{2\pi i}\oint_\Gamma\frac{dz}{z-H(x)}, \tag{1.4}$$

where the contour Γ circles $\lambda(x)$ counterclockwise in the complex z-plane. In the examples we consider here levels cross at $x=0$. On $\mathbf{R}^n/\{0\}, P(x)$ inherits the smoothness of $H(x)$, and in particular has fixed dimension. The adiabatic evolution transports vectors from the range of $P(x)$ to the range of $P(x')$. More precisely:

The *adiabatic connection* $A(P)$ is the operator-valued one-form[1]

$$A(P)(x)=-[(dP)(x),P(x)]. \tag{1.5}$$

[1] This definiton differs from the one of [6] by a factor of i