A METHOD OF REGULARIZING THE EQUATIONS
OF MOTION IN THE CENTRAL FORCE-FIELD

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Abstract. This paper deals with a method of regularization and linearization of the equations of motion in the central force-field, when the potential is given.

This method of regularization of the equations of motion is known (Sundman, 1913), and is based on the transformation of time by means of introducing a new independent variable.

In this article a condition has been obtained for the regularizing function when the potential is given. Some examples of the perturbed Keplerian motions are discussed.

0. Introduction

Recently there have appeared papers aimed at regularizing the equations of motion by means of introducing some new variable in such a way that: firstly, the (pole type) singularity arising is eliminated and, secondly, trying to linearize the equations of motion by making them the equations of harmonic oscillators and including the cases with imaginary frequency.

These papers [1–5] in general have been devoted to various aspects of Kepler's problem and the regularization given has proved to be fairly effective on considering perturbed Kepler's motion.

For conservative systems with \( n \) degrees of freedom the problems of regularizing the equations of motion have been considered in [6, 7].

This paper deals with the problem of regularizing the equations of motion in the central force-field, when the potential is given, by means of introducing a regularizing transformation of time.

The idea of such a method of regularization is connected with the well known work of Sundman [8].

1. The Fundamental Relations

We shall consider the motion of a particle \( M \) of reduced mass \( \mu \) in a central force field, when the potential \( V(r) \) and the constant \( h \) of the total energy are given.

The Hamiltonian \( H \) of the system in the plane of the orbit can be written in polar coordinates \( r \) (distance) and \( \varphi \) (angle) in the well known form [9]

\[
H = \frac{1}{2\mu} \left( p_r^2 + \frac{1}{r^2} p_\varphi^2 \right) + V(r) \quad (H(p, q) = h). \tag{1.1}
\]
Using the cyclic integral \( P_\phi = \mu r^2 \dot{\phi} = c \) (\( c \) is the constant of angular momentum) and the impulse \( P_r = \mu \dot{r} \) we obtain the following expression for the energy integral

\[
\frac{1}{2} \mu \left( \frac{\mathrm{d}r}{\mathrm{d}t} \right)^2 + \frac{\mu c^2}{2r^2} + V(r) = h
\]  

(1.2)

or

\[
\frac{1}{2} \mu \left( \frac{\mathrm{d}r}{\mathrm{d}t} \right)^2 = h - V_1(r)
\]  

(1.3)

where \( V_1(r) = V(r) + \mu c^2/2r^2 \) – the so called 'effective potential energy' [10].

When the potential \( V(r) \) of the force field is given, the qualitative analysis of the Equation (1.3) may be realized by Weierstrass' method [11].

Another method based on time-transformation will be given below.

2. Regularization of Time

A new independent variable \( \tau = \tau(t) \) called 'fictitious time', is introduced by means of the differential relation

\[
\mathrm{d}\tau = g^{-1}(r) \, \mathrm{d}t
\]  

(2.1)

where \( g(r) \) is a properly chosen function of \( r \) which belongs to the class \( c^1 \) and does not vanish in the domain of the phase-space in which the initial system is being considered.

Then Equation (1.3) can be written in the following form

\[
\frac{1}{2} \mu \left( \frac{\mathrm{d}r}{\mathrm{d}\tau} \right)^2 = g^2(r) (h - V_1(r)).
\]  

(2.2)

Differentiating of this equation with respect to \( \tau \) and dividing by the unequal to zero factor \( \mathrm{d}r/\mathrm{d}\tau \) we finally obtain

\[
\mu \frac{\mathrm{d}^2r}{\mathrm{d}\tau^2} = (h - V_1(r)) \frac{\mathrm{d}g^2(r)}{\mathrm{d}r} - g^2(r) \frac{\mathrm{d}V_1(r)}{\mathrm{d}r}.
\]  

(2.3)

In order to regularize the equation of motion with the aim of its linearization, we must require the right-hand side of (2.3) to be a linear function of \( r \).

This condition gives the linear differential equation of the first order relatively to the regularizing function \( g^2(r) \)

\[
(h - V_1(r)) \frac{\mathrm{d}g^2(r)}{\mathrm{d}r} - \frac{\mathrm{d}V_1(r)}{\mathrm{d}r} \cdot g^2(r) = c_1 r + c_2.
\]  

(2.4)

\((c_1, c_2 = \text{const.})\)

The singular points of this equation are roots of the equation \( V_1(r) - h = 0 \), i.e. those points in which the radial velocity \( v_r = \mathrm{d}r/\mathrm{d}t \) vanishes in accordance with (1.3).