KRASNOSEL'SKII NUMBERS FOR BOUNDED FINITELY STARLIKE SETS IN $R^d$

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For each $k$ and $d$, $1 \leq k \leq d$, define $f(d,d) = d + 1$ and $f(d,k) = 2d$ if $1 \leq k \leq d - 1$. The following results are established:

Let $\mathcal{M}$ be a uniformly bounded collection of compact, convex sets in $R^d$. For a fixed $k$, $1 \leq k \leq d$, $\dim \{M : M \in \mathcal{M}\} \geq k$ if and only if for some $\alpha > 0$, every $f(d,k)$ members of $\mathcal{M}$ contain a common $k$-dimensional set of measure (volume) at least $\alpha$.

Let $S$ be a bounded subset of $R^d$. Assume that for some fixed $k$, $1 \leq k \leq d$, there exists a countable family of $(k-1)$-flats $\{H_i : i \geq 1\}$ in $R^d$ such that $\text{cl } S \sim S \subset \bigcup \{H_i : i \geq 1\}$ and for each $i \geq 1$, $(\text{cl } S \sim S) \cap H_i$ has $(k-1)$ dimensional measure zero. Every finite subset of $S$ sees via $S$ a set of positive $k$-dimensional measure if and only if for some $\alpha > 0$, every $f(d,k)$ points of $S$ see via $S$ a set of $k$-dimensional measure at least $\alpha$.

The numbers of $f(d,d)$ and $f(d,1)$ above are best possible.

INTRODUCTION.

We begin with some definitions from [2]. Let $S$ be a set in $R^d$. For points $x$ and $y$ in $S$, we say $x$ sees $y$ via $S$ ($x$ is visible from $y$ via $S$) if and only if the corresponding segment $[x,y]$ lies in $S$. Set $S$ is called finitely starlike if and only if every finite subset $F$ of $S$ sees via $S$ a common point, and the subset of $S$ seen by $F$ is called the $F$-star of $S$. Finally, $S$ is starshaped if and only if there is some point $p$ in $S$ such that $p$ sees via $S$ each point of $S$, and the set of all such points $p$ is the (convex) kernel of $S$.

A familiar theorem by Krasnosel'skii [7] states that for $S$ a nonempty compact set in $R^d$, $S$ is starshaped if and only if every $d+1$ points of $S$ see via $S$ a common point. The Krasnosel'skii theorem fails for noncompact sets. However, many of the sets for which it fails turn out to be finitely starlike instead of starshaped. This observation has lead Peterson [10] to conjecture that a Krasnosel'skii-type theorem exists to characterize finitely starlike sets, too. Furthermore, since

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there exist Krasnosel'skii-type theorems which yield the dimension of the kernel of a starshaped set ([1], [3]), analogues of these results could exist for the dimension of $F$-stars in finitely starlike sets.

Earlier work [2] has given some answers in the planar case by placing restrictions on the set $(\text{int cl } S) \sim S$. For example, when $(\text{int cl } S) \sim S$ lies in a countable union of lines and has 1-dimensional measure zero, then every 4 point subset of $S \subseteq \mathbb{R}^2$ sees via $S$ a set of positive 2-dimensional Lebesgue measure if and only if every finite subset of $S$ sees via $S$ such a set. In this paper, we obtain related results for $d$-dimensional sets.

The following terminology will be used throughout the paper: conv $S$, aff $S$, cl $S$, int $S$, rel int $S$, and ker $S$ will denote the convex hull, affine hull, closure, interior, relative interior, and kernel, respectively, for set $S$. Dim $S$ will be the dimension of the affine hull of $S$. In case $S \subseteq \mathbb{R}^d$ is Lebesgue measurable in some $k$-flat, $1 \leq k \leq d$, then $\mu_k(S)$ will represent its $k$-dimensional Lebesgue measure. For distinct points $x$ and $y$, $R(x, y)$ will be the ray emanating from $x$ through $y$. Finally, dist will denote the Euclidean metric for $\mathbb{R}^d$.


THE RESULTS.

We begin with the following analogue of [3, Lemma].

THEOREM 1. For each $k$ and $d$, $1 \leq k \leq d$, define $f(d, d) = d + 1$ and $f(d, k) = 2d$ if $1 \leq k \leq d - 1$. Let $M$ be a uniformly bounded collection of compact convex sets in $\mathbb{R}^d$. Then for a fixed $k$ with $1 \leq k \leq d$, $\dim \cap \{M : M \in M\} \geq k$ if and only if for some $\alpha > 0$, every $f(d, k)$ members of $M$ contain a common $k$-dimensional set of measure (volume) at least $\alpha$.

Proof. Before beginning the argument, observe that if certain members of $M$ contain a $k$-dimensional set $A$ having measure at least $\alpha$, then these members also contain the convex, measurable set $\text{cl conv } A$, and $\text{cl conv } A$ has measure at least $\alpha$. Moreover, the $k$-dimensional measure of $\text{cl conv } A$ is exactly its $k$-dimensional volume.

Now we begin the proof. The necessity of the condition is obvious. To establish the sufficiency, we start with the case for $k = d$. Then by the remarks above, every $f(d, d) = d + 1$ members of $M$ intersect in a set of $d$-dimensional volume at least $\alpha$. Hence by a result of Polikanova and Perel'man [11], $\cap \{M : M \in M\}$ has positive $d$-dimensional volume, finishing the argument.

To prove the result for arbitrary $k$, $1 \leq k \leq d$, we use an argument similar to one in [3, Lemma]. We proceed by induction on the dimension of our space. If $d = 1$, then $k = 1$, and the argument is finished by the proof for $k = d$. Inductively, assume the result is true for natural numbers less than $d$, $2 \leq d$, to prove for $d$. If $k = d$, again the argument is finished, so assume $1 \leq k < d$. Then every $f(d, k) = 2d$ members of $M$ contain a common $k$-dimensional set of measure at least $\alpha$. 