FIRST AND SECOND-ORDER NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS FOR INFINITE-DIMENSIONAL PROGRAMMING PROBLEMS

H. Maurer
Universität Münster, W. Germany

J. Zowe
Universität Würzburg, W. Germany

14 April 1977
Received 14 April 1977
Revised manuscript received 7 July 1978

First-order and second-order necessary and sufficient optimality conditions are given for infinite-dimensional programming problems with constraints defined by arbitrary closed convex cones. The necessary conditions are immediate generalizations of those known for the finite-dimensional case. However, this does not hold for the sufficient conditions as illustrated by a counterexample. Here, to go from finite to infinite dimensions, causes an essential change in the proof-techniques and the results. We present modified sufficient conditions of first-order and of second-order which are based on a strengthening of the usual assumptions on the derivative of the objective function and on the second derivative of the Lagrangian.

Key words: Mathematical Programming, Optimality Conditions, Lagrange-Multipliers, Banach Spaces.

1. Introduction

Let $f$ be a functional defined on a real Banach space $X$, $g$ a map from $X$ into a real Banach space $Y$ and $K$ a closed convex cone in $Y$. We consider the mathematical programming problem

$$
\begin{align*}
\text{minimize} & \quad f(x), \\
\text{subject to} & \quad g(x) \in K.
\end{align*}
$$

(P)

A point $\bar{x} \in K$ is called optimal for (P) if $g(\bar{x}) \in K$ and if $f$ restricted to $g^{-1}(K)$ assumes a local minimum at $\bar{x}$. Throughout this paper we assume that the first and second Fréchet-derivatives $f'(\bar{x})$, $g'(\bar{x})$ and $f''(\bar{x})$, $g''(\bar{x})$ at the considered $\bar{x}$ exist. The maps $f''(\bar{x})$ and $g''(\bar{x})$ are interpreted as bilinear forms on $X \times X$. We will derive conditions of Kuhn–Tucker type for the first and second derivative of $f$ and $g$ at $\bar{x}$ which are necessary and sufficient, respectively, for the optimality of $\bar{x}$.

Necessary optimality conditions of first and of second-order for (P) and various extensions and specializations of (P) are well-known in the literature; cf. e.g. [2, 3, 7, 8, 9, 11, 13] and the survey articles [1, 6]. In this paper we merely
relax one assumption made in [11] for the proof of the Kuhn–Tucker theorem. For first and second-order necessary conditions, the proof techniques and the conclusions are the same for the finite-dimensional and the infinite-dimensional situation. However, this changes completely when trying to establish sufficient optimality conditions. Here the standard proof techniques for the finite-dimensional case (it suffices that $X$ is of finite dimension, $Y$ may be arbitrary) uses in a decisive way the compactness of the unit sphere in $X$, cf. e.g. [3, 4]. Since this does not hold for infinite-dimensional $X$ the arguments known for the finite-dimensional case do not carry over to the infinite-dimensional problem. The approaches of Borwein [1], Ioffe and Tikhomirov [5], which give sufficient conditions for the infinite-dimensional situation, do not face this difficulty by imposing additional assumptions. Borwein [1] gives first-order sufficient conditions under convexity assumptions. Also, he derives second-order sufficient conditions by assuming that the critical part of the unit sphere of $X$ is weakly compact. Ioffe and Tikhomirov [5] establish second-order sufficient conditions for equality constraints only.

We do not make any such restrictive assumptions on either the space $X$ or on the type of the constraints. This leads to the surprising fact that the sufficient conditions known for the finite-dimensional case have to be considerably strengthened for infinite-dimensional spaces; this is demonstrated by an example. We give direct proofs for our modified sufficient conditions and show that our conditions reduce to the known conditions for finite-dimensional $X$.

The paper is organized as follows. In Section 2 we discuss some regularity assumptions used later on. Necessary conditions and a Kuhn–Tucker theorem are given in Section 3. Then we prove an approximation property for the feasible set which is needed in the demonstration of our sufficient conditions. Finally, in Section 5, we present our modified sufficient optimality conditions.

In the finite-dimensional case, second-order sufficient conditions play an important role when studying differentiable perturbations of $(P)$ or the convergence behaviour of iterative algorithms for finding optimal points for $(P)$. Our hope is that the second-order sufficient conditions given in Theorem 5.6 may be useful for similar investigations of the infinite-dimensional problem.

### 2. Regularity conditions

The set of the feasible points for $(P)$ is denoted by $M$, i.e., $M = g^{-1}(K)$. The following two cones which approximate $M$ at a given $\bar{x} \in M$ will be important for the formulation of our optimality conditions; cf. Kurcyusz [6].

$$T(M, \bar{x}) = \{ h \in X \mid h = \lim(x_n - \bar{x})/t_n, x_n \in M, t_n > 0, t_n \to 0 \},$$

$$L(M, \bar{x}) = \{ h \in X \mid g'(\bar{x}) h \in K_{g(\bar{x})} = g'(\bar{x})^{-1}(K_{g(\bar{x})}) \};$$

(2.1)