GENERALIZED POLYMATROIDS AND
SUBMODULAR FLOWS

András FRANK and Éva TARDOS

Eötvös University Budapest, Mathematical Institute, Department of Computer Science, Műzeum körút 6–8, Budapest VIII, Hungary 1088

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Polyhedra related to matroids and sub- or supermodular functions play a central role in combinatorial optimization. The purpose of this paper is to present a unified treatment of the subject. The structure of generalized polymatroids and submodular flow systems is discussed in detail along with their close interrelation. In addition to providing several applications, we summarize many known results within this general framework.

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1. Introduction

The first relationship between matroid theory and what is now called combinatorial optimization was a theorem by Rado (1942) on the existence of independent transversals of a family of sets. As a next significant contribution Rado (1957) proved that the greedy algorithm works correctly not only on graphs but on matroids as well.

In the middle of the sixties investigations by J. Edmonds (1965a) and (1965b) further emphasized the role of matroids in combinatorial optimization. Edmonds’s matroid intersection and matroid partition theorems along with Rado’s theorem became prototypes of matroid min–max theorems. These three results are somehow on the same level in the sense that they can be derived from each other by elementary constructions. The weighted matroid intersection problem of J. Edmonds, seems to be on a higher level. Edmonds also developed polynomial-time algorithms for the matroid partition and for the weighted matroid intersection problems.

A second fundamental idea is the use of linear programming in combinatorial optimization. The idea goes back to works of Dantzig, Ford, Fulkerson and Hoffman, who applied linear programming to derive combinatorial results concerning networks. Later, Edmonds realized that linear programming can also be used in cases when the constraint matrix corresponding to the combinatorial problem is not necessarily totally unimodular.

The principle of using linear programming is nowadays rather well-known: Associate points in $R^n$ with combinatorial objects to be investigated, determine the linear inequalities describing the convex hull $P$ of these points, and apply the linear programming duality theorem in order to obtain a min–max result for the optimal object.

Matroid polyhedra were amongst the first polyhedra defined this way by Edmonds (1971). The independent sets of a matroid are the combinatorial objects to be investigated. The convex hull of their incidence vectors defines the matroid polyhedron. Edmonds (1971) showed that the matroid polyhedron is described by \( \{ x \in R^n : x(A) \leq r(A) \text{ for every } A \subseteq S \} \), where $r$ is the rank function and $S$ the ground set of the matroid. A much deeper result of Edmonds (1970) establishes the polyhedron of common independent sets of two matroids.

As a natural generalization of matroid polyhedra Edmonds (1970) introduced polymatroids. A fundamental feature of polymatroids is that the optimum of a linear objective function over a polymatroid can be calculated by a greedy algorithm. Furthermore, the defining linear system is totally dual integral (TDI). Edmonds also established the polymatroid intersection theorem (1970) stating, roughly, that