Spectral Representations of Lorentz Invariant Distributions and Scale Transformation*

A. RIECKERS and W. GÜTTINGER
Department of Physics, University of Munich

Received June 15, 1967

Abstract. An approach to the theory of Lorentz invariant distributions is developed in terms of covariant spectral representations. The behaviour of singular invariant distributions under a change of scale is analyzed. It is shown that the conventional extension of homogeneous singular functions into distributions in $R^4$, followed by a breakdown of homogeneity, is incomplete. Homogeneous extensions depending on an arbitrary scaling parameter are introduced, calculation techniques are developed and various formulae having applications in quantum field theory are derived.

1. Introduction

The aim of this paper is to present a new approach to Lorentz invariant distributions in terms of spectral representations which exhibit the covariant form of the functionals, permit to overcome the "origin of the light cone" difficulties and lead to a considerable simplification of calculation techniques. Thus, the approach is possibly simpler than that of Refs. [1]—[5]. On the basis of this formalism we investigate the behaviour of certain singular invariant distributions under a change of scale: The conventional way of associating distributions in $R^4$ with homogeneous singular functions by regularization gives rise to functionals which are no longer homogeneous. Such inhomogeneous distributions (e.g. the propagators $(x^2 - i0)^{-n}, n \geq 2$) are physically unacceptable because the space-time or momentum variables on which they depend carry dimension. The extension of singular homogeneous functions into homogeneous distributions in $R^4$ requires the introduction of an arbitrary scaling parameter bearing dimension. Then, a breakdown of dilatation symmetry by regularization can be avoided if a similarity transformation in space-time is accompanied by a corresponding change of this scaling parameter.

In Sec. 2 we develop the theory of Lorentz invariant distributions in terms of spectral representations. In Sec. 3 we analyze the problem

---

* Supported by the Deutsche Forschungsgemeinschaft.
of extending distributions from a subspace to the entire $R^4$ in connection with scale transformations. Sec. 4 is devoted to a discussion of delta-functions $\delta^{(n)}(x^2)$ etc. localized on the light cone and to a comparison of the various definitions appearing in the literature, all of them being summed up into an expression depending on the arbitrary scaling parameter. In Sec. 5 asymptotic and Laurent expansions are derived and the equivalence of certain extensions is proved. After a brief discussion of some special algebraically singular distributions in Sec. 6 we study in Sec. 7 Fourier transforms and analytic functionals in terms of spectral representations.

2. Lorentz Invariant Distributions in Terms of Spectral Representations

We denote by $R^4$ the four-dimensional Minkowski space of real points $x = (x_0, x, x_1, x_2, x_3)$ with the metric $x^2 = x_0^2 - x_0x - x_i^2$. When we write $f(x)$ for a distribution $f \in \mathcal{D'}(R^4)$ we merely wish to indicate that $f$ operates on test functions $\varphi(x)$ depending on $x \in R^4$. $\langle \varphi \rangle$ denotes the value of the distribution at the element $\varphi(x) = \varphi(x_0, x)$.

**Definition.** A distribution $f \in \mathcal{D'}(R^4)$ is said to be invariant under the restricted Lorentz group $L^+_4$ if for any $A \in L^+_4$ the relation

$$A f(x) \langle \varphi \rangle = f(Ax) \langle \varphi(Ax) \rangle = f(x) \langle \varphi(x) \rangle$$

holds for all $\varphi \in \mathcal{D}(R^4)$.

Such $f$ will be called "invariant". In terms of the infinitesimal generators of $L^+_4$, $M_{0k} = x_k \partial / \partial x_0 + x_i \partial / \partial x_k - x_k \partial / \partial x_i$, $(i, k = 1, 2, 3)$ $f$ is invariant only if $M_{0k} f = M_{ik} f = 0$.

**Definition.** Let $A^*$ be an element of the antichronous component $L^+_{4\ast}$. Then we define the reflected distribution $\tilde{f}$ by

$$\tilde{f} = A^* f.$$  

For every invariant $f$, e.g. $\tilde{f}(x) = f(-x)$, the definition (2.2) is independent of the particular choice of $A^*$. A distribution $f$ is called even if $\tilde{f} = f$ and odd if $\tilde{f} = -f$.

Every invariant distribution $f$ can be decomposed into an even part $f_e$ and an odd part $f_o$ according to

$$f = f_e + f_o$$

where

$$f_e = \frac{1}{2} (f + \tilde{f}), \quad f_o = \frac{1}{2} (f - \tilde{f}).$$

From Ref. [6] we recall the following theorem.