ON A MATROID DEFINED BY EAR-DECOMPOSITIONS OF GRAPHS

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A. Frank described in [1] an algorithm to determine the minimum number of edges in a graph G whose contraction leaves a factor-critical graph and he asked if there was an algorithm for the weighted version of the problem. We prove that the minimal critical-making edge-sets form the bases of a matroid and hence the matroid greedy algorithm gives rise to the desired algorithm.

1. Introduction

Given a connected graph G, what is the minimum number of edges whose contraction leaves a factor-critical graph? A. Frank [1] noticed that for 2-edge-connected graphs this value equals the minimum number $\varphi(G)$ of even ears in ear-decompositions of G, and he proved a minimax formula for $\varphi(G)$. In the same paper he proposed the problem of describing the structure of the edge-sets above. The aim of this note is to prove that minimal critical-making sets form the bases of a matroid. We refer the reader to [3] for basic concepts of matroids.

For a connected graph G, an edge-set is called critical-making if its contraction leaves a factor-critical graph. A graph G is factor-critical if for every $v \in V(G)$, $G - v$ has a perfect matching. Since factor-critical graphs are 2-edge-connected, every cut edge of G is contained in any critical-making edge-set. Thus we may assume that G is 2-edge-connected.

Let $G = (V, E)$ be an undirected, 2-edge-connected graph. An ear-decomposition of G is a sequence $G_0, G_1, \ldots, G_n = G$ of subgraphs of G where $G_0$ is a vertex and each $G_i$ arises from $G_{i-1}$ by adding a path $P_i$ for which the two end-vertices (they are not necessarily distinct) belong to $G_{i-1}$ while the inner vertices of $P_i$ do not. This means the graph G can be written in the following form: $G = P_1 + P_2 + \ldots + P_n$ where the paths $P_i$ are called the ears of this decomposition. An ear is odd (resp. even) if its length is odd (resp. even). Let us consider an ear-

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decomposition of $G$ which has as few even ears as possible. Let $\varphi(G)$ denote this minimum number of even ears.

The \textit{contraction} of an edge $e$ of a graph $G$ is defined in the usual way. We will denote the contracted graph by $G/e$. Note that the contraction of an edge can produce parallel edges. By the contraction of an edge-set $F$ of $G$ we mean the graph $G' = G/F$ arising from $G$ by contracting each edge of $F$. When we contract an edge-set then we will always assume (without loss of generality) that this edge-set is circuit-free, that is, it is a forest. By contracting a connected subgraph $H$ of $G$ we mean the contraction of a spanning tree of $H$.

The \textit{subdivision} of an edge-set $F$ of a graph $G$ means that we subdivide each edge $e$ of $F$ by a new vertex. The resulting graph is denoted by $G \times F$. The following lemma gives the relation between contraction and subdivision.

\textbf{Lemma 1.1.} Let $G$ be a 2-edge-connected graph and let $k$ be a positive integer. Then the following are equivalent:

a) the minimum number of edges whose contraction leaves a factor-critical graph is $k$,

b) the minimum number of edges whose subdivision leaves a factor-critical graph is $k$,

c) the minimum number of even ears in an ear-decomposition of $G$ is $k$, i.e. $\varphi(G) = k$.

The proof of this lemma is given in Section 2 and it implies that $\varphi(G/F) = \varphi(G \times F)$ for any forest $F$. In the view of this fact we shall use the subdivision of an edge-set rather than the contraction. (It is easier to deal with subdivision than with contraction.) We would like to emphasize that Lemma 1.1 is not completely trivial. It is not true that the ear-decomposition of $G/e$ can always be extended to an ear-decomposition of $G$ (see Figure 1). For more details see [6, Lemma 9.2].

![Fig. 1](image-url)

Clearly, the subdivision of any edge in a graph either decreases or increases $\varphi(G)$ by one. Thus $\varphi(G \times F) \geq \varphi(G) - |F|$ for any edge-set of $G$. An edge-set $F$ of a graph $G$ is called \textit{ear-extreme} if $\varphi(G \times F) = \varphi(G) - |F|$. Note that every ear-extreme edge-set is a forest. Our purpose is to prove that the ear-extreme edge-sets form the independent sets of a matroid. Clearly, the ear-extreme edge-sets of maximum size and the critical-making sets of minimum size are the same. Furthermore, by