An approximation algorithm for the maximum independent set problem is given, improving the best performance guarantee known to $O(n/(\log n)^2)$. We also obtain the same performance guarantee for graph coloring. The results can be combined into a surprisingly strong simultaneous performance guarantee for the clique and coloring problems.

The framework of subgraph-excluding algorithms is presented. We survey the known approximation algorithms for the independent set (clique), coloring, and vertex cover problems and show how almost all fit into that framework. We show that among subgraph-excluding algorithms, the ones presented achieve the optimal asymptotic performance guarantees.

1. Introduction.

An independent set in a graph is a set of vertices with no edges connecting them. The problem of finding an independent set of maximum size is one of the classical $\mathcal{NP}$-hard problems. We consider polynomial time algorithms that find an independent set that is not necessarily optimal, but of a guaranteed size. The quality of the approximation is given by the ratio of the size of the maximum independent set to the size of the approximation found, and the largest such ratio over all inputs gives the performance guarantee of the algorithm.
A few other problems are closely related to the independent set problem. A *clique* is a set of mutually connected vertices. Since finding the maximum size clique in a graph is equivalent to finding the maximum independent set in the complement of the graph, the clique problem is for our purposes the same problem.

A *vertex cover* is a set of vertices with the property that every edge in the graph is incident to some vertex in the set. Note that vertices not in a given vertex cover must be independent, hence finding a maximum independent set is equivalent to finding a minimum vertex cover. Approximations to the two problems, however, differ widely.

The third related problem is *graph coloring*, namely finding an assignment of as few colors as possible to the vertices so that no adjacent vertices share the same color. Because the colors induce a partition of the graph into independent sets, the problems of approximating independent set and coloring are closely related. The dual problem to graph coloring is finding a *clique cover*, which is a partition of the graph into disjoint cliques.

The analysis of approximation algorithms for graph coloring started with Johnson [17] who showed that the greedy algorithm colors $k$-colorable graphs with $O(n/\log_k n)$ colors, obtaining a performance guarantee of $O(n/\log n)$. Several years later, Wigderson [21] introduced an elegant algorithm that colors $k$-colorable graphs with $O(kn^{1/(k-1)})$ colors, which, when combined with Johnson's result, yields an $O(n(\log \log n/\log n)^2)$ performance guarantee. Recently, Berger and Rompel [4] presented an algorithm that improves on Johnson's idea to obtain an $O(n/(\log \log n)^2)$ coloring. When combined with Wigderson's method, they obtain an $O(n(\log n)^3)$ performance guarantee. Halldórsson [16] improved that to $O(n(\log n)^2)$,

Finally, Blum has improved the best ratio for small values of $k$, in particular for 3-coloring from the $O(\sqrt{n})$ of Wigderson and the $O(\sqrt{n/\log n})$ of Berger and Rompel, to $n^{0.4+o(1)}$ [6] and later to $n^{0.375+o(1)}$ [7].

We shall present an efficient graph coloring algorithm that colors $k$-colorable graphs with $O(n^{(k-2)/(k-1)})$ colors when $k \leq 2 \log n$, and $O((\log n/\log (k/\log n))$ when $k \geq 2 \log n$. The algorithm strictly improves on both Johnson's and Wigderson's method.

Folklore (see [15, p. 134] attributed to Gavril) tells us that any maximal matching approximates the minimum vertex cover by a factor of two. This was slightly improved independently by Bar-Yehuda and Even [3], and Monien and Speckenmeyer [19], to a factor of $2 - \Omega(\log \log n/\log n)$, but no further improvements have been found.

Approximating the independent set problem has seen less success. No approximation algorithm yielding a non-trivial performance guarantee has been found in the literature. One of the main results of this paper is an algorithm that obtains an $O(n/(\log n)^2)$ performance guarantee for the independent set problem on general graphs, as well as several results on graphs with a high independence number.