UNIQUENESS RESULTS AND ALGORITHM FOR STACKELBERG–COURNOT–NASH EQUILIBRIA

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Abstract

This paper uses recently developed theory of sensitivity analysis to explore the reaction of a Cournot–Nash equilibrium to a Stackelberg firm, and to analyze the effect of this reaction on the uniqueness of the Stackelberg–Cournot–Nash equilibrium. Some of the results presented are not new, but the methods used provide simpler proofs and a different perspective. More importantly, the methods used here allow the development of new conditions for a unique Stackelberg–Cournot–Nash equilibrium that extends those previously known. The methods used also provide for the development of an efficient algorithm for finding the equilibrium.

1. Introduction

Several years ago, Sherali, Soyster and Murphy [3] characterized Stackelberg–Cournot–Nash equilibria and presented an algorithm for computing such equilibria. A Stackelberg–Cournot–Nash equilibrium is one in which there are \( N + 1 \) firms, \( N \) of which – the follower or Cournot firms – optimize their production quantities given the other firms' production quantities, under the assumption that the other firms' quantities will not change in response. The \((N + 1)\)st firm, the leader or Stackelberg firm, optimizes its production quantities taking into account the reactions of the other \( N \) firms to its production decisions. The Stackelberg–Cournot–Nash equilibrium problem is of interest because the Cournot–Nash model serves as the basis for models of oligopolistic homogeneous product markets, such as energy products and natural resources, in which market price is determined by supply; modeling the Stackelberg firm in these markets is useful for the analysis of strategic behavior. However, for our purposes it is also of interest in that it is one of the simpler realizations of a mathematical program with equilibrium constraints. As such, it can shed some light on the behavior of these problems, and techniques developed for this problem may be useful for more complex mathematical programs with equilibrium constraints.

In particular, these problems generally are not convex programs, or at least it can not be shown that they are. However, in many cases, the reaction of the constraining equilibrium problem is not large relative to the changes in the decision
variables. If the objective function is strictly concave, and the implicit function defined by the equilibrium problem does not have large curvature, then the resulting problem will still have a unique solution. This may explain the surprisingly good results obtained for many of these problems using heuristics that can only find local optima. See, for example, Suwansirikul, Friesz and Tobin [4] and Friesz, Tobin and Cho [1].

The purpose of this paper is to use recently developed theory of sensitivity analysis to explore the reaction of the Cournot–Nash equilibrium to the Stackelberg decision variable and the analyze the effects of this reaction on the uniqueness of the Stackelberg–Cournot–Nash equilibrium. Some of the results presented here have been presented in Sherali et al., but the methods used here provide simpler proofs and a different perspective. More importantly, the methods used here allow the development of new conditions for a unique Stackelberg–Cournot–Nash equilibrium that extend those developed in Sherali et al. They also provide the tools for the development of an efficient algorithm for finding the equilibrium. It is hoped that the method of analysis used here will also be useful in the study of more complex equilibrium constrained mathematical programming problems.

In section 2, a mathematical formulation of the problem being studied is given. In section 3, sensitivity analysis results for Cournot–Nash equilibria are used to characterize the reaction of the Cournot–Nash equilibrium to the Stackelberg decision variable. This characterization is used in section 4 to develop sufficient conditions for the Stackelberg–Cournot–Nash equilibrium to be unique. In section 5, an example that illustrates the robustness of the uniqueness of a Stackelberg–Cournot–Nash equilibrium is discussed, and an algorithm for finding the equilibrium is presented.

2. Cournot–Nash and Stackelberg–Cournot–Nash equilibria

Let there be $N$ profit-maximizing firms, $i = 1, \ldots, N$, which produce the same product. Let $p(Q), Q \geq 0$, denote the inverse demand function (price function), where $Q$ is the total production in the market. Let $q_i \geq 0$ denote the $i$th firm's production and let $q = [q_1, \ldots, q_N]$ be the vector of production amounts for all firms. Therefore,

$$Q = \sum_{i=1}^{N} q_i,$$

and let

$$\bar{Q}_i = \sum_{j=1}^{N} q_j.$$

Finally, let $c_i(q_i)$ be the $i$th firm's total cost of producing $q_i$ units. The profit for firm $i$ is given by