Matrix Representation and Gradient Flows for NP-Hard Problems

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Abstract. Over the past decade, a number of connections between continuous flows and numerical algorithms were established. Recently, Brockett and Wong reported a connection between gradient flows on the special orthogonal group $SO(n)$ and local search solutions for the assignment problem. In this paper, we describe a uniform formulation for certain NP-hard combinatorial optimization problems in matrix form and examine their connection with gradient flows on $SO(n)$. For these problems, there is a correspondence between the so-called 2-opt solutions and asymptotically stable critical points of an associated gradient flow.

Key Words. Gradient flows, assignment problem, traveling salesman problem, graph partitioning problem, local search.

1. Introduction

Over the past decade, a number of connections between continuous flows and numerical algorithms were established; see, for example, Bayer and Lagarias (Ref. 1), Bloch (Ref. 2), Brockett (Refs. 3 and 4), Brockett and Wong (Ref. 5), Chu (Ref. 6), Deift, Nanda, and Tomei (Ref. 7), Faybusovich (Refs. 8 and 9), Symes (Refs. 10 and 11), and Watkins and Elsner (Ref. 12). Included in this list, among many others, are the classical connection between the Toda lattice flow and the QR algorithm, the relation of affine and projective trajectories with the Karmarkar algorithm and its variants, and the connection between gradient flows on the special orthogonal group and the least squares optimization problem.

These results are of interest to a variety of audiences. From the system theory perspective, other than their potential of leading to advances in...
computation, they provide many novel classes of dynamic systems with interesting structures and properties, such as the so-called double bracket equations introduced in Ref. 3.

In this paper, we will describe a uniform formulation for certain NP-hard combinatorial optimization problems in matrix form and examine their connection with gradient flows on the space \( \mathcal{SO}(n) \) of the special orthogonal matrices.

In Ref. 13, Karmarkar proposed using a steepest descent approach to solve combinatorial optimization problems; see also Refs. 14–16. Independently in Ref. 5, the gradient flow approach was applied to a class of combinatorial optimization problems known as the assignment problem. One of the key results reported in that paper is that, for any given assignment problem, there is a simple correspondence between local minima of a certain local search algorithm and local minima of an associated gradient flow defined on \( \mathcal{SO}(n) \).

Since the assignment problem is known to have a polynomial time solution (Ref. 17), an algorithm to find a local minimum does not have great significance. In this paper, however, we show that the \( \mathcal{SO}(n) \) gradient flow approach can be extended easily to a large class of combinatorial optimization problems that includes the traveling salesman problem and the graph partitioning problem. Since these problems are well known to be NP-hard, a local search algorithm is commonly used to solve approximately these problems. Thus, results connecting local search algorithms for these problems with gradient flows may have a more practical significance. Moreover, the problem of finding a local minimum may be computationally complex itself. The complexity issue of local search algorithms is an interesting topic first raised by Johnson, Papadimitriou, and Yannakakis (Ref. 18). In particular, for the graph partitioning problem with the SWAP neighborhood (two partitions are neighbors if one can be made identical to the other by swapping two vertices), it was shown by Schäffer and Yannakakis (Ref. 19) that finding the local minimum from an arbitrary initial point is NP-hard. One of our theorems here shows that the set of local minima for the graph partitioning problem with the SWAP neighborhood has a simple correspondence with the asymptotically stable critical points of an associated gradient flow on \( \mathcal{SO}(n) \).

The extension of the gradient flow approach to these NP-hard problems is based on the observation that any combinatorial optimization problem is representable as an optimization problem on the set of permutation matrices. Such a matrix form representation for many of the commonly known combinatorial optimization problems takes on a very simple form, and in the case of the traveling salesman problem is probably well known in the folklore. In the first part of this paper, we will present a general discussion of the