THE CAUCHY PROBLEM FOR A SEMILINEAR WAVE EQUATION. I

L. V. Kapitanskii

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The Cauchy problem for a semilinear wave equation on the torus $T^n$, $n \geq 3$:

$$
\begin{align*}
\ddot{u} - \Delta u + f(u) &= h, & u \big|_{t=0} = \psi, & \dot{u} \big|_{t=0} = \varphi, \\
\end{align*}
$$

is investigated. It is assumed that the function $f : \mathbb{R}^r \to \mathbb{R}^r$ is continuous and there exist nonnegative constants $A_1$, $A_2$, $A_3$, and $a \geq 1$ such that

$$
\begin{align*}
A_1 + A_2 s^a + \int_0^1 f(t) dt \geq 0, & \quad \forall s \in \mathbb{R}^r, \\
|f(s_1) - f(s_2)| \leq A_3 (|s_1|^{a-1} + |s_2|^{a-1}) |s_1 - s_2|, & \quad \forall s_1, s_2 \in \mathbb{R}^r.
\end{align*}
$$

It is proved that if the parameter $a$ lies in the interval $1 \leq a < (n + 2)/(n - 2)$, then for any $q \in W_2^1(T^n)$, $\psi \in L_2(T^n)$, $h \in L_1, \text{loc}(\mathbb{R}^1 \to L_2(T^n))$ the problem (1) has a unique solution $u$, global with respect to time, such that $u \in C_{\text{loc}}(\mathbb{R}^1 \to W_2^1(T^n))$, $\dot{u} \in C_{\text{loc}}(\mathbb{R}^1 \to L_2(T^n))$ and $u \in L_q, \text{loc}(\mathbb{R}^1 \to L_p(T^n))$ for all $p, q$ satisfying the relations

$$
\frac{n-3}{4n} < \frac{1}{p} < \frac{n-2}{4n}, \quad \frac{q}{q-1} = \frac{n-2}{n-p}.
$$

INTRODUCTION

In this paper we investigate the problem of the existence and uniqueness of a solution, global with respect to time, of the Cauchy problem for the semilinear wave equation on the torus $T^n \times \mathbb{R}^n/\mathbb{Z}^n$, $n \geq 3$:

$$
\begin{align*}
\begin{cases}
\ddot{u} - \Delta u + f(u) &= h, & \quad t \in \mathbb{R}^1, & x \in \mathbb{T}^n, \\
\dot{u} \big|_{t=0} &= \psi, & \ddot{u} \big|_{t=0} &= \varphi,
\end{cases}
\end{align*}
$$

where a dot over $u$ denotes differentiation with respect to $t$, $\Delta$ is the usual Laplace operator, $h$ is an arbitrary function of the variables $(t, x)$, satisfying the condition

$$
\begin{align*}
h \in L_1([t_1, t_2] \to L_4(T^n)), & \quad \forall t_1, t_2,
\end{align*}
$$

the initial data $\psi$, $\varphi$ are taken from the spaces

$$
\begin{align*}
\psi \in W_2^1(T^n), & \quad \varphi \in L_4(T^n).
\end{align*}
$$

Regarding the function $f$, defining the nonlinearity, we assumed that $f : \mathbb{R}^r \to \mathbb{R}^r$ is continuous and there exist nonnegative constants $A_1$, $A_2$, $A_3$, and $a \geq 1$ such that

$$
\begin{align*}
A_1 + A_2 s^a + \int_0^1 f(t) dt \geq 0, & \quad \forall s \in \mathbb{R}^r, \\
|f(s_1) - f(s_2)| \leq A_3 (|s_1|^{a-1} + |s_2|^{a-1}) |s_1 - s_2|, & \quad \forall s_1, s_2 \in \mathbb{R}^r.
\end{align*}
$$

We mention that under these assumptions we have

$$
|f(s)| \leq A_4 (|s|^{a})
$$

with some constant $A_n \geq 0$, so that the number $a$ defines the order of growth of $|f(s)|$ for $|s| \to +\infty$.

By a generalized solution (or, simply, solution) of the Cauchy problem (1) on the time interval $[0, T]$ we mean a function $u$ such that

$$u \in L_{a}(\mathbb{T}^n), \quad \dot{u} \in L_{a}(\mathbb{T}^n),$$

the function $f(u(\cdot))$ is summable on $[0, T] \times \mathbb{T}^n$, the integral identity

$$\int_0^T \int_{\mathbb{T}^n} (-\dot{u} \eta + \nabla u \cdot \nabla \eta + f(u) \eta - h \eta) dt dx - \int_{\mathbb{T}^n} \eta(0) dx = 0$$

holds for each smooth function $\eta$, equal to zero for $t \geq T$, and the initial condition $u(0, x) = \varphi(x)$ holds for a.a. $x \in \mathbb{T}^n$. The function $u : \mathbb{R} \times \mathbb{T}^n \to \mathbb{R}$ is called a global (with respect to time) solution of problem (1) if it is a solution on each finite interval $[0, T]$. [In a similar manner one determines the solutions for $t < 0$. We consider the solutions only for $t \geq 0$ since, in order to obtain the solution for $t < 0$, it is sufficient to invert the time direction in (0.1), (0.2) and to solve again the corresponding Cauchy problem for $t \geq 0$.]

It is well known that when the dimension $n$ of the space is equal to 1 or 2, the problem (1) has a unique global solution for any $h$, $\varphi$, $\psi$ satisfying (0.3), (0.4), provided $f$ is subjected to the restrictions (0.5) and (0.6) with an arbitrary $a \geq 1$; therefore, in the sequel we shall assume that $n \geq 3$.

We formulate the fundamental result of the present paper.

**THEOREM 1.** Assume that $f$ satisfies the conditions (0.5) and (0.6) with some $a$ from the interval $1 \leq a < (n + 2)/(n - 2)$. Then

1) for any $\psi$, $\varphi$, and $h$, satisfying (0.3), (0.4), the problem (1) has a unique global solution $u$ such that

$$u \in L_{a}(\mathbb{T}^n), \quad \dot{u} \in L_{a}(\mathbb{T}^n), \quad 0 \leq t < \infty,$$

for all $p_n$ and $q_n$, satisfying the relations

$$\frac{k - 2}{4n} < \frac{1}{p_n} < \frac{n - 2}{2n}, \quad \frac{1}{q_n} = \frac{k - 2}{4} - \frac{k}{p_n};$$

2) this solution has the following smoothness property with respect to $t$:

$$u \in C([0, T] \to W_{2}^{a}(\mathbb{T}^n), \quad \dot{u} \in C([0, T] \to L_{a}(\mathbb{T}^n), \quad 0 \leq t < \infty.$$

The proof of Theorem 1 is given in Secs. 2 and 3.

Here it is appropriate to make several remarks. It is known that, under the assumptions (0.3)-(0.6), for all $a \geq 1$ there exists at least one global solution of the problem (1) (see [1]), where the existence has been proved even under restrictions on $f$, weaker than (0.6), admitting a super power growth of $|f(s)|$ for $|s| \to \infty$. It is easy to show, following, for example the proof of Theorem 1.2 of [2], that if $a$ lies in the interval $1 \leq a \leq n/(n - 2)$, then the solution is unique. Here the fundamental problem is the uniqueness of the solution in the case $a > n/(n - 2)$. For $a \geq (n + 2)/(n - 2)$ this problem remains entirely open.

Apparently, so far, in the literature devoted to initial-boundary-value problems for the semilinear wave equation, the case $a > n/(n - 2)$ has been considered only in [3]. Moreover, in [3] one has considered strong or even classical solutions of an initial-boundary-value problem for equations of type (0.1) (with an elliptic operator of order $2m$ instead of $A$ and with $h \equiv 0$). In the theorems of [3], on the function $f$ one imposes rather strict restrictions, but if one traces the admissible order $f$ of the growth of $|f(s)|$ when $|s| \to \infty$, then $a$ has to be smaller than $[(n + 2)/(n - 2) - \nu_n]$, where $\nu_n \geq 0$ depends on the dimension $n$ and on the smoothness of the considered solutions and, moreover, $\nu_n$ is trivially greater than zero for $n > 9$. 

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