The structural information content $I_g(X)$ of a graph $X$ was treated in detail in three previous papers (Mowshowitz 1968a, 1968b, 1968c). Those investigations of $I_g$ point up the desirability of defining and examining other entropy-like measures on graphs. To this end the chromatic information content $I_c(X)$ of a graph $X$ is defined as the minimum entropy over all finite probability schemes constructed from chromatic decompositions having rank equal to the chromatic number of $X$. Graph-theoretic results concerning chromatic number are used to establish basic properties of $I_c$ on arbitrary graphs. Moreover, the behavior of $I_c$ on certain special classes of graphs is examined. The peculiar structural characteristics of a graph on which the respective behaviors of the entropy-like measures $I_c$ and $I_g$ depend are also discussed.

1. Introduction. In this paper we will discuss an entropy measure $I_c$ defined with respect to a class of chromatic decompositions of a finite undirected graph. First, we will examine the behavior of this measure on arbitrary finite undirected graphs, and then specialize to particular cases. Second, we will compare $I_c$ with $I_g$; finally, we will discuss the significance of the notion of graphical information content and summarize our results.

We begin with some definitions.* A homomorphism of a graph $X$ into a graph $Y$ is a mapping $\phi$ from $V(X)$ into $V(Y)$ such that whenever $[x, y] \in E(X)$, $[x, y] \phi = [x\phi, y\phi] \in E(Y)$. An equivalent way of defining this notion is to define an elementary homomorphism of a graph $X$ as the identification of two non-adjacent points; then a homomorphism is just a sequence of elementary

* The definitions given here are largely those of Hedetniemi (1966).

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ENTROPY AND THE COMPLEXITY OF GRAPHS: IV.
ENTROPY MEASURES AND GRAPHICAL STRUCTURE

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The structural information content $I_g(X)$ of a graph $X$ was treated in detail in three previous papers (Mowshowitz 1968a, 1968b, 1968c). Those investigations of $I_g$ point up the desirability of defining and examining other entropy-like measures on graphs. To this end the chromatic information content $I_c(X)$ of a graph $X$ is defined as the minimum entropy over all finite probability schemes constructed from chromatic decompositions having rank equal to the chromatic number of $X$. Graph-theoretic results concerning chromatic number are used to establish basic properties of $I_c$ on arbitrary graphs. Moreover, the behavior of $I_c$ on certain special classes of graphs is examined. The peculiar structural characteristics of a graph on which the respective behaviors of the entropy-like measures $I_c$ and $I_g$ depend are also discussed.

1. Introduction. In this paper we will discuss an entropy measure $I_c$ defined with respect to a class of chromatic decompositions of a finite undirected graph. First, we will examine the behavior of this measure on arbitrary finite undirected graphs, and then specialize to particular cases. Second, we will compare $I_c$ with $I_g$; finally, we will discuss the significance of the notion of graphical information content and summarize our results.

We begin with some definitions.* A homomorphism of a graph $X$ into a graph $Y$ is a mapping $\phi$ from $V(X)$ into $V(Y)$ such that whenever $[x, y] \in E(X)$, $[x, y] \phi = [x\phi, y\phi] \in E(Y)$. An equivalent way of defining this notion is to define an elementary homomorphism of a graph $X$ as the identification of two non-adjacent points; then a homomorphism is just a sequence of elementary

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homomorphisms. \( \phi \) is called a full homomorphism of \( X \) into \( Y \) if \([x\phi, y\phi] \in E(Y)\) implies that there exist points \( u, v \in V(X) \) such that \( x\phi = u\phi \), \( y\phi = v\phi \) and \([u, v] \in E(X)\). The image of \( X \) under the homomorphism \( \phi \) is the graph \( X\phi \), with \( V(X\phi) = \{x\phi \mid x \in V(X)\} \) and \( E(X\phi) = \{[x\phi, y\phi] \mid [x, y] \in E(X)\} \). Clearly, \( X\phi \subseteq Y \) if \( \phi \) is a homomorphism of \( X \) into \( Y \); moreover, \( X\phi \) is a section subgraph of \( Y \) if \( \phi \) is a full homomorphism. If \( \phi \) maps \( V(X) \) onto \( V(Y) \), then \( \phi \) is called a homomorphism of \( X \) onto \( Y \). Note that if \( \phi \) is a full homomorphism of \( X \) onto \( Y \), then \( E(X)\phi = E(Y) \); if, in addition, \( \phi \) is one-one, then \( \phi \) is an isomorphism. A homomorphism \( \phi \) is said to be of order \( n \) if \( n = |V(X\phi)| \), and is complete of order \( n \) if \( X\phi \cong K_n \).

A coloring of a graph \( X \) is an assignment of colors to the points of \( X \) such that no two adjacent points have the same color. An \( n \)-coloring of \( X \) is a mapping \( f \) of \( V(X) \) onto the set \( \{1, 2, \ldots, n\} \) such that whenever \([x, y] \in E(X)\), \( xf \neq yf \), that is a coloring of \( X \) which uses \( n \) colors. An \( n \)-coloring \( f \) is complete if for every \( i, j \) with \( i \neq j \) there exist adjacent points such that \( xf = i \) and \( yf = j \). A decomposition \( \{V_i\}_{i=1}^n \) of the set \( V(X) \) of points of \( X \) is said to be a chromatic decomposition of \( X \), if \( x, y \in V_i \) imply that \([x, y] \notin E(X)\). Clearly, if \( f \) is an \( n \)-coloring of \( X \), the sets \( \{x \in V(X) \mid xf = i\} \) for \( i = 1, 2, \ldots, n \) form a chromatic decomposition of \( X \); conversely, a chromatic decomposition \( \{V_i\}_{i=1}^n \) determines an \( n \)-coloring \( f \). Thus, the sets \( V_i \) are called color classes. The chromatic number \( \kappa(X) \) is the smallest number \( n \) for which \( X \) has an \( n \)-coloring, or, equivalently, the smallest \( n \) for which \( X \) has a chromatic decomposition with \( n \) color classes. Note that a graph \( X \) can have more than one \( n \)-coloring (or chromatic decomposition with \( n \) color classes). \( X \) is called \( n \)-chromatic if \( \kappa(X) = n \).

The following remarks concerning the relationship between homomorphisms and \( n \)-colorings (illustrated in Fig. 1) are necessary for the sequel. It is easy to show (Hedetniemi, 1966, 10) that a graph \( X \) has a complete \( n \)-coloring \( f \) if and only if there exists a complete homomorphism \( \phi \) of \( X \) onto the complete graph \( K_n \). From this it follows that if \( \kappa(X) = n \), then \( X \) has a complete homomorphism of order \( n \); and that the smallest order of all homomorphisms of a graph \( X \) is just the chromatic number \( \kappa(X) \). Thus, it is clear that to each chromatic decomposition \( \{V_i\}_{i=1}^n \) of an \( n \)-chromatic graph \( X \), there corresponds a homomorphism \( \phi \) of \( X \) onto \( K_n \) such that each \( V_i \) is of the form

\[
\{x\phi = u \mid x \in V(X)\}
\]

for some \( u \in V(K_n) \).

2. The chromatic information content of a graph. Since the automorphism group of a graph \( X \) gives rise to a unique decomposition of \( V(X) \), we were able