The locally testable semigroups were discovered in the study of finite automata (see [8], [12] for history and motivation). In this paper, we study the locally testable semigroups from a purely algebraic viewpoint, as a simultaneous generalization of both the normal bands and the nilpotent semigroups. We generalize the results of [8] and [12] to the infinite case and, under mild restrictions on the semigroups, obtain subdirect product decompositions which sharpen these results.

DEFINITION. Let $S$ be a semigroup. $S^+$ will denote the set of all finite sequences of elements of $S$. Let $k$ be a positive integer. If $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$, $m, n \geq k$ are elements of $S^+$ such that

\[
\begin{align*}
(x_1, \ldots, x_k) &= (y_1, \ldots, y_k) \\
(x_{m-k+1}, \ldots, x_m) &= (y_{n-k+1}, \ldots, y_n) \\
\{(x_i, \ldots, x_{i+k-1}) : 2 \leq i \leq m-k\} &= \{(y_j, \ldots, y_{j+k-1}) : 2 \leq j \leq n-k\}
\end{align*}
\]

we say that $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ have the same k-test vectors. $S$ is k-testable iff for $u = (x_1, \ldots, x_m)$ and $v = (y_1, \ldots, y_n)$ in $S^+$, $m, n \geq k$, if $u$ and $v$ have the same k-test vectors then $x_1x_2\cdots x_m = y_1y_2\cdots y_n$ (products in $S$). It is easy to see that a k-testable semigroup is m-testable, for all $m \geq k$. $S$ is locally testable semigroups.

* This research was supported in part by the Advanced Research Projects Agency of the Office of the Secretary of Defense (F44620-70-C-0107) and is monitored by the Air Force Office of Scientific Research. A preliminary version of this paper appeared as a technical report, Department of Computer Science, Carnegie-Mellon University, March 1971.

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testable iff \( S \) is \( k \)-testable, for some \( k > 0 \). Let \( S \) be locally testable. The degree of \( S \) is the smallest \( k \) for which \( S \) is \( k \)-testable. Let \( \text{LTS} \) be the class of all locally testable semigroups.

Let \( S \) be a semigroup. \( E(S) \) will denote the set of idempotents of \( S \). \( S \) divides \( T \) iff \( S \) is a homomorphic image of a subsemigroup of \( T \).

A semigroup \( S \) is nil iff \( S \) has a zero element \( 0 \) and for all \( s \) in \( S \), there is a positive integer \( n \) (depending on \( s \)) such that \( s^n = 0 \). A semigroup \( S \) is nilpotent iff \( S \) has a zero element \( 0 \) and there exists a positive integer \( n \) such that \( S^n = 0 \). The smallest such \( n \) is called the degree of nilpotence of \( S \). As is well known (see e.g. [1], p. 179), a finite semigroup is nil iff it is nilpotent. \( S \) is a nilpotent extension of an ideal \( I \) iff the Rees factor semigroup \( S/I \) is nilpotent.

A semigroup is locally finite iff all its finitely generated subsemigroups are finite. A semigroup \( S \) is periodic iff every element of \( S \) has a finite period. A finite semigroup \( S \) is combinatorial iff it has only trivial subgroups or equivalently iff there is a positive integer \( n \) such that for all \( x \) in \( S \), \( x^{n+1} = x^n \). All undefined terminology and notation follows [2].

Our first proposition shows how to construct large classes of locally testable semigroups. In fact, we will see below (Theorem 1) that all finite locally testable semigroups can be constructed this way.

**Proposition 1 ([12])**

(a) A rectangular band is 1-testable. A completely 0-simple semigroup having trivial subgroups is 2-testable.

(b) \( \text{LTS} \) is closed under finite direct products, subsemigroups, homomorphisms, antihomomorphisms and nilpotent extensions.

(c) The class of \( k \)-testable semigroups is closed under arbitrary direct products, subsemigroups, homomorphisms and antihomomorphisms.