ALL UNCOUNTABLE CARDINALS
CAN BE SINGULAR

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ABSTRACT
Assuming the consistency of the existence of arbitrarily large strongly compact cardinals, we prove the consistency with ZF of the statement that every infinite set is a countable union of sets of smaller cardinality. Some other statements related to this one are investigated too.

0. Introduction

It is well known that the axiom of choice implies that a countable union of countable sets is countable and that for every ordinal $\alpha \mathcal{N}_{\alpha+1}$ is a regular cardinal. Without the axiom of choice the picture is quite different.

Levy [6] proved that it is consistent with ZF that $\mathcal{N}_1$ is singular. This leads naturally to the problem of generalizing Levy’s result to all $\mathcal{N}_\alpha$’s. The purpose of this paper is to show that such a generalization is possible if one is ready to allow for the consistency of some large cardinals.

The exact statement of our main theorem is:

THEOREM 1. If $\text{ZFC} + (\forall \alpha \in 0^n)((\exists k > \alpha) (k \text{ is a strongly compact cardinal}))$ is consistent then $\text{ZF} + (\forall \alpha \in 0^n)(\text{cf} \mathcal{N}_\alpha = \mathcal{N}_0)$ is consistent too.

It is known that the consistency of $\text{ZF} + (\forall \alpha \in 0^n)(\text{cf} \mathcal{N}_\alpha = \mathcal{N}_0)$ cannot be proved without assuming the consistency of the existence of some large cardinals. Jensen’s Covering Theorem [1] implies that if both $\mathcal{N}_1$ and $\mathcal{N}_2$ are singular then $0^\#$ exists. Recently Jensen and Dodd [2] were able to show that under the same assumption, one can obtain an inner model with a measurable cardinal. The large cardinals assumption which we use here is much stronger. Even to make only $\mathcal{N}_1$ and $\mathcal{N}_2$ singular we need a cardinal $k$ which is $\lambda$-strongly compact for every $\lambda < k^{++}$, where $k^{+0} = k$, $k^{+(n+1)} = (k^{+n})^+$ and $k^{++} = \bigcup_{n<\omega} k^{+n}$.

Received November 24, 1978 and in revised form May 15, 1979
Following Specker [10], let us consider the class \( \Omega = \{ \alpha \in \mathbb{On} \mid (\forall \beta \leq \alpha) (\beta = 0 \lor (\exists \gamma) (\beta = \gamma + 1) \lor \text{there is a sequence} \langle \gamma_n \mid n < \omega \rangle \text{ such that for each } n \gamma_n < \beta \text{ and } \beta = \bigcup_{n<\omega} \gamma_n \rangle ) \).

As Specker has shown exactly one of the following alternative hold.

1. \( \Omega = \mathbb{On} \),
2. \( \Omega = \omega_\alpha \),
3. \( \Omega = \omega_{\alpha+1} \), where \( \alpha \) is an ordinal.

Levy and Feferman [6] constructed a model in which \( \Omega = \omega_2 \). Levy [7] obtained some interesting consequences of (3). If \( \Omega \supseteq \omega_3 \), then Jensen and Dodd's surprising results in [2] show that there is a measurable cardinal in some inner model. Using some large cardinal assumptions, we prove that each one of the alternatives (1)–(3), for any ordinal \( \alpha \), can hold. Using the same constructions we get the following:

**Theorem II.** If \( \text{ZFC} + (\forall \alpha \in \mathbb{On}) (\exists k > \alpha) (k \text{ is a strongly compact cardinal}) \) is consistent, so are

(a) \( \text{ZF} + \text{ 'every infinite set is a countable union of sets of smaller cardinality' } \)

(b) \( \text{ZF} + \Omega = \mathbb{On} + \text{ 'there is an uncountable set } A \text{ such that for every sequence } \langle A_i \mid i \in I \rangle, \text{ which is increasing with respect to inclusion, if } |A_i| < |A| \text{ for every } i \in I \text{ and } A = \bigcup_{i \in I} A_i, \text{ then } |I| \geq |A| \rangle \).

Let us give a brief outline of the proof of Theorems I and II. We first extend the universe \( V \) by a filter \( G \) generic over a proper class of forcing conditions which are similar to those of Prikry [8]. In the universe \( V[G] \) thus obtained all regular cardinals of \( V \), and hence all cardinals, become ordinals of cofinality \( \omega \). \( V[G] \) satisfies all the axioms of ZFC except for the power set axiom. Inside \( V[G] \) we construct a symmetric model \( N_G \) by the method described in Jech [4, 5]. The well ordered cardinals of \( N_G \) are exactly \( \omega \), all the strongly compact cardinals of \( V \) and their limits. Every such cardinal has cofinality \( \omega \) in \( N_G \). Also, in this model every infinite set is a countable union of sets of smaller cardinality. To obtain a model which satisfies (b) of Theorem II, we add by forcing an amorphous set \( A \) to this model (i.e. a set \( A \) such that if \( B \subseteq A \) then \( |B| < \aleph_0 \) or \( |A - B| < \aleph_0 \)).

I am grateful to E. A. Palujtin for supervision of part of the work. I would like to thank Professor Thomas Jech for pointing out errors in an earlier version. Most of all, I wish to express my gratitude to Professor Azriel Levy and Professor Menachem Magidor, for the many fruitful conversations we had and for their patience and their suggestions during the time this work was being completed.