Mediterr. J. Math. 9 (2012), 453–485
DOI 10.1007/s00009-011-0141-9
1660-5446/12/030453-33, *published online* April 29, 2011

Mediterranean Journal of Mathematics

A Filippov's Theorem, Some Existence Results and the Compactness of Solution Sets of Impulsive Fractional Order Differential Inclusions

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Abstract. In this paper, we first present an impulsive version of Filippov's Theorem for fractional differential inclusions of the form,

$$\begin{array}{rcl} D_*^{\alpha}y(t) & \in & F(t,y(t)), \quad \text{a.e. } t\in J\backslash\{t_1,\dots,t_m\},\ \alpha\in(0,1],\\ y(t_k^+)-y(t_k^-) & = & I_k(y(t_k^-)), \quad k=1,\dots,m,\\ y(0) & = & a, \end{array}$$

where $J=[0,b]$, D_*^{α} denotes the Caputo fractional derivative and F is a set-valued map. The functions I_k characterize the jump of the solutions at impulse points t_k ($k=1,\dots,m$). In addition, several existence results are established, under both convexity and nonconvexity conditions on the multivalued right-hand side. The proofs rely on a nonlinear alternative of Leray-Schauder type and on Covitz and Nadler's fixed point theorem for multivalued contractions. The compactness of solution sets is also investigated.

Mathematics Subject Classification (2010). 34A60, 34A37.

Keywords. Fractional differential inclusions, fractional derivative, fractional integral, Caputo fractional derivatives,impulsive differential inclusions.

1. Introduction

Differential equations with impulses were considered for the first time in the 1960's by Milman and Myshkis [43, 44]. A period of active research, primarily in Eastern Europe from 1960-1970, culminated with the monograph by Halanay and Wexler [27].

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The dynamics of many evolving processes are subject to abrupt changes, such as shocks, harvesting and natural disasters. These phenomena involve short-term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of an entire evolution. In models involving such perturbations, it is natural to assume these perturbations act instantaneously or in the form of "impulses". As a consequence, impulsive differential equations have been developed in modeling impulsive problems in physics, population dynamics, ecology, biotechnology, industrial robotics, pharmacokinetics, optimal control, and so forth. Again, associated with this development, a theory of impulsive differential equations has been given extensive attention. Works recognized as landmark contributions include [8, 39, 51, 55]. There are also many different studies in biology and medicine for which impulsive differential equations are good models (see, for example, [3, 36, 37] and the references therein).

In recent years, many examples of differential equations with impulses with fixed moments have flourished in several contexts. In the periodic treatment of some diseases, impulses correspond to administration of a drug treatment or a missing product. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs.

During the last ten years, impulsive ordinary differential inclusions and functional differential inclusions with different conditions have been intensely studied by many mathematicians. At present the foundations of the general theory are already laid, and many of them are investigated in detail in the books of Aubin [4], Benchohra *et al* [9] and Henderson and Ouahab [30] and the references therein.

Differential equations with fractional order have recently proved valuable tools in the modeling of many physical phenomena [19, 23, 24, 40, 41]. There has been a significant theoretical development in fractional differential equations in recent years; see the monographs of Kilbas *et al* [33], Miller and Ross [42], Podlubny [52], Samko *et al* [54], and the papers of Bai and Lu [7], Diethelm *et al* [18–20], El-Sayed and Ibrahim [21], Kilbas and Trujillo [34], Mainardi [40], Momani and Hadid, [45], Momani *et al* [46], Nakhushev [48], Podlubny *et al* [53], and Yu and Gao [57].

Very recently, some basic theory for initial value problems for fractional differential equations and inclusions involving the Riemann-Liouville differential operator was discussed by Benchohra *et al* [10], Lakshmikantham [38]. El-Sayed and Ibrahim [21] initiated the study of fractional multivalued differential inclusions.

Applied problems require definitions of fractional derivatives allowing a utilization that is physically interpretable for initial conditions containing $y(0)$, $y'(0)$, etc. The same requirements are true for boundary conditions. Caputo's fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types, see Podlubny [52].