Stability of the Blok Theorem

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Abstract. W. Blok proved that varieties of boolean algebras with a single unary operator uniquely determined by their class of perfect algebras (i.e., duals of Kripke frames) are exactly those which are intersections of conjugate varieties of splitting algebras. The remaining ones share their class of perfect algebras with uncountably many other varieties. This theorem is known as the Blok dichotomy or the Blok alternative. We show that the Blok dichotomy holds when perfect algebras in the formulation are replaced by $\omega$-complete algebras, atomic algebras with completely additive operators or algebras admitting residuals. We also generalize the Blok dichotomy for lattices of varieties of boolean algebras with finitely many unary operators. In addition, we answer a question posed by W. Dziobiak and show that classes of lattice-complete algebras or duals of Scott-Montague frames in a given variety are not determined by their subdirectly irreducible members.

1. Introduction

One of the most fascinating results obtained by Wim Blok was a complete characterization of degrees of Kripke incompleteness of normal modal logics [2]. The theorem, known as the Blok alternative or the Blok dichotomy, can be expressed in the algebraic language as follows: a variety $V$ of boolean algebras with a single normal unary operator is either uniquely determined by its class of duals of Kripke frames (perfect algebras) or there are uncountably many other varieties which cannot be distinguished from $V$ this way. The main goal of the present research is to examine how relevant the restriction to perfect algebras is, i.e., whether the theorem holds when duals of Kripke frames in the formulation are replaced by other classes of algebras. Below, we sketch some background, but the reader may feel the need for a more detailed introduction. Chagrov and Zakharyaschev [6] (see also [36] and [35]) provide all the necessary mathematical information on modal logic, Venema [33] explains the connection between modal logic and algebra, Goldblatt [13] describes the development of the field, while Rautenberg et al. [23] gives an account of the motivation behind the Blok theorem itself, the troubled history of its publication and its influence on the modal logic community.
1.1. Motivation and historical background. We are going to work with normal modal logics based on classical propositional calculus. Algebraically, these systems correspond to boolean algebras with normal operators (BAOs); throughout the paper, we are assuming that all operators are unary and there are only finitely many of them. The concept of an operator on a boolean algebra was introduced by Jónsson and Tarski [16, 17] (see also Jónsson [15]). It is an operation which distributes over finite joins on every coordinate. An operator is normal if it preserves bottom; throughout the paper, we restrict attention to normal operators. If, in addition, the operator distributes over all existing (not only finite) joins, it is called completely additive. One of the main results of Jónsson and Tarski was an extension of Stone’s representation theorem for boolean algebras. They proved that every BAO can be embedded into a perfect algebra, i.e., a lattice-complete and atomic BAO where all operators are completely additive. The class of perfect algebras will be denoted as $\mathcal{CAV}$. A variety $V$ is called (CAV-)complete if it is of the form $V = HSP(V \cap \mathcal{CAV})$, i.e., if it is generated as a variety from a set of $\mathcal{CAV}$'s.

Jónsson and Tarski [16, 17] proved that $\mathcal{CAV}$'s are exactly those BAOS which arise as dual algebras of relational structures. As relational structures in modal logic came to be known as Kripke frames, perfect algebras are also sometimes called Kripke algebras. This duality between $\mathcal{CAV}$'s and frames can be, in fact, extended to the level of (complete) morphisms to obtain a full-blown category-theoretical duality—see Remark 2.19 below. This explains the use of the notion completeness above: a variety is $\mathcal{CAV}$-complete iff the corresponding logic is Kripke complete (see any of the references mentioned at the beginning of the paper for definitions of logical notions appearing henceforth). Thomason [28] showed that there is a nontrivial variety of boolean algebras with two conjugated unary operators which does not contain any perfect algebra, i.e., there are no Kripke frames adequate for the corresponding logic. It follows from Makinson’s Theorem (cf. Theorem 3.1 below) that there is no such variety of boolean algebras with one unary operator, i.e., modal algebras. Nevertheless, Fine [11] found an example of an incomplete variety of closure algebras (or $\mathbf{S4}$-algebras), that is, modal algebras satisfying all the equations which hold for the closure operator $\text{Int}$ on the powersets of arbitrary topological spaces.

The final section of Fine’s paper posed a problem, which in algebraic terms may be formulated as follows. If a variety $V$ of boolean algebras with $n$ operators—strictly speaking, Fine was interested in the case of modal algebras—is not complete, there is another variety $V' \neq V$ satisfying the following “equation”:

$$V' \cap \mathcal{CAV} = V \cap \mathcal{CAV}$$

(cav-inc)

For example, the largest complete subvariety $V' = HSP(V \cap \mathcal{CAV})$ of $V$ satisfies cav-inc. More generally, we can say $V'$ is $\mathcal{CAV}$-equivalent to $V$ if cav-inc holds. If