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Symmetries on the Lattice of k-Bounded Partitions* †

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Abstract. In 2002, Suter [25] identified a dihedral symmetry on certain order ideals in Young's lattice and gave a combinatorial action on the partitions in these order ideals. Viewing this result geometrically, the order ideals can be seen to be in bijection with the alcoves in a 2-fold dilation in the geometric realization of the affine symmetric group. By considering the m-fold dilation we observe a larger set of order ideals in the k-bounded partition lattice that was considered by Lapointe, Lascoux, and Morse [14] in the study of k-Schur functions. We identify the order ideal and the cyclic action on it explicitly in a geometric and combinatorial form.

Keywords: affine reflection groups, symmetry

1. Introduction

For each $k \in \mathbb{N}$, Suter described a set Y^k of partitions with unexpected dihedral symmetries [25]. The set Y^k is an order ideal in Young's lattice, and so to specify Y^k it is enough to specify that its maximal elements are exactly those rectangles with hook-lengths at most k.

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† This research was facilitated by computer exploration using the open-source mathematical software `Sage` [21] and its algebraic combinatorics features developed by the `Sage-Combinat` community [27].

Definition 3.1. ([25]) *Let* $R_i := \left(i^{k+1-i}\right)$ *for* $1 \leq i \leq k$. *Then*

$$Y^k := \{\lambda : \lambda \subseteq R_i, \text{ for some } 1 \leq i \leq k\}.$$

Suter proved that Y^k has the same symmetries as the affine Dynkin diagram of type $\widetilde{\mathcal{A}}_k$ by explicitly describing a cyclic action on $\lambda \in Y^k$ which, along with conjugation of a partition, gives a dihedral action on Y^k.

Since any $\lambda \in Y^k$ has maximum hook-length at most k, λ may be viewed as a k-bounded partition $\big($or $(k+1)$-core$\big)$, so that Y^k can be equivalently described as an order ideal in the lattice of k-bounded partitions (defined in Section 2.3). It is well known that the dominant alcoves in type $\widetilde{\mathcal{A}}_k$ are indexed by k-bounded partitions.

Theorem 2.1. ([16, 19]) *There is an order-preserving bijection between the lattice of k-bounded partitions, the lattice of* $(k+1)$*-cores, and weak order on the dominant alcoves in type* $\widetilde{\mathcal{A}}_k$.

We can therefore associate to each partition $\lambda \in Y^k$ a dominant alcove A_λ. More precisely, Suter showed in [26] that Y^k is in bijection with alcoves in $2A_\emptyset$, the two-fold dilation of the fundamental alcove $A_\emptyset$ in type $\widetilde{\mathcal{A}}_k$. As the fundamental alcove has a $(k+1)$-fold cyclic symmetry, so does $2A_\emptyset$ — and so does Y^k. Thus, the natural geometric symmetry on $2A_\emptyset$ explains the unexpected symmetry of Y^k.

The bijection between Y^k and $2A_\emptyset$ sends the maximal elements $R_i \in Y^k$ to the alcoves $A_{R_i} \in 2A_\emptyset$ with a facet on the hyperplane $H_{\alpha_0,2} = \{x\colon \langle \alpha_0, x\rangle = 2\}$ (where α_0 is the highest root) and a single vertex on the hyperplane $H_{\alpha_0,1} = \{x\colon \langle \alpha_0, x\rangle = 1\}$. Such an alcove is characterized by the unique fundamental weight Λ_i that is a vertex of A_{R_i} not on the hyperplane $H_{\alpha_0,2}$. We emphasize this with correspondence with a proposition.

Proposition 3.2. *The map* $R_i \mapsto \Lambda_i$ *is a bijection between the maximal elements of* Y^k *and the fundamental weights.*

This first motivation is reviewed in more detail in Section 3.

Our second motivation comes from k-Schur functions, which — like the dominant alcoves in type $\widetilde{\mathcal{A}}_k$ — are also indexed by $(k+1)$-cores λ or k-bounded partitions μ. Proposition 3.2 has an algebraic analogue in the theory, whereby the k-Schur function s_{R_i} may be expressed as a sum over the $\mathcal{A}_k$-orbit of the fundamental weight Λ_i. These rectangles R_i also appear in Lapointe and Morse's paper [17], where they prove the following theorem.

Theorem 4.4. ([17, Theorem 40]) *For a rectangle* R_i *and a k-bounded partition* μ,

$$s_\mu^{(k)} s_{R_i}^{(k)} = s_{\mu \cup R_i}^{(k)},$$

where $\mu_1 \cup \mu_2$ *denotes the partition obtained by combining the parts of* μ_1 *and* μ_2 *and placing them into non-increasing order.*

In Section 5, we consider products of the form

$$s^{(k)}_{\cup_{j=1}^{m-1} R_{i_j}} = \prod_{j=1}^{m-1} s^{(k)}_{R_{i_j}},$$