On Schwarz-type smoothers for saddle point problems with applications to PDE-constrained optimization problems

René Simon · Walter Zulehner

Abstract In this paper we consider a (one-shot) multigrid strategy for solving the discretized optimality system (KKT system) of a PDE-constrained optimization problem. In particular, we discuss the construction of an additive Schwarz-type smoother for a certain class of optimal control problems. A rigorous multigrid convergence analysis is presented. Numerical experiments are shown which confirm the theoretical results.

Mathematics Subject Classification (2000) 65N22 · 65N55 · 49M15

1 Introduction

In this paper we discuss multigrid methods for solving large-scale systems of discretized mixed variational problems. The main applications considered here are optimization problems in function spaces with constraints in form of partial differential equations (PDEs). The necessary (and for the problems considered here also sufficient) first-order optimality conditions on a solution of such a problem can be written as a mixed variational problem, usually called the optimality system or Karush–Kuhn–Tucker (KKT) system.

The work was supported by the Austrian Science Fund (FWF) under grant SFB 013/F1309.

R. Simon (✉)
SFB 013, Johannes Kepler University Linz, Altenbergerstraße 69, 4040 Linz, Austria
e-mail: rene.simon@sfb013.uni-linz.ac.at

W. Zulehner
Institute of Computational Mathematics, Johannes Kepler University, Altenbergerstraße 69, 4040 Linz, Austria
e-mail: zulehner@numa.uni-linz.ac.at
In particular, we will consider elliptic optimal control problems, see, e.g., \cite{15,19}. In such problems the primal unknown, say \( x \), consists of two parts: a function \( y \), the so-called state, and a function \( u \), the so-called control. The problem is to find \( x = (y, u) \) from appropriate function spaces that minimizes a given cost functional subject to a constraint, the so-called state equation, which, for each control \( u \), is an elliptic boundary value problem in \( y \). The corresponding KKT system involves another (dual) unknown, say \( p \) (the Lagrangian multiplier or the adjoint state), and consists of three components: the state equation (see above), the adjoint state equation, which, for each state \( y \), is an elliptic boundary value problem in \( p \), and the control equation, which is typically an algebraic relation between \( u \) and \( p \).

In principle, there are two different approaches for mixed problems, such as KKT systems, to take advantage of the multigrid idea. One way is to use an outer iteration, typically a preconditioned Richardson method (possibly accelerated by a Krylov subspace method), applied to the discretized mixed problem. For typical preconditioners of KKT systems in elliptic optimal control, see, e.g., \cite{3–5,14} and the references cited there. These preconditioners usually rely on efficient solvers or preconditioners for the state equation (as a PDE in \( y \)) and the adjoint state equation (as a PDE in \( p \)) and on the construction of a good preconditioner for the corresponding Schur complement of the KKT system, which is the reduced Hessian of the Lagrangian. A preconditioner based on a different Schur complement is proposed in \cite{17}. Multigrid techniques (as an inner iteration or approximation) can be used for (some or all of) these components, see, e.g., \cite{11,12,17}.

The other way is to use multigrid methods directly applied to the (discretized) mixed problem as an outer iteration based on appropriate smoothers (as a sort of inner iteration). For PDE-constrained optimization problems this approach is also known as one-shot multigrid strategy, see \cite{18}. One of the most important ingredients of such a multigrid method is an appropriate smoother.

A first approach for constructing such smoothers is to combine standard smoothers applied to the components elliptic state and adjoint equations complemented with a special relaxation method for the control equation, see, e.g., \cite{1}.

A second class of smoothers are point smoothers, where the variables, here \( y, u \) and \( p \), are grouped pointwise (with respect to the points (nodes) of the underlying mesh) and one or several sweeps of point-block Jacobi or point-block Gauss–Seidel sweeps with respect to this grouping are performed, see, e.g, \cite{6}.

A natural extension of point smoothers are patch smoothers: The computational domain is divided into small (overlapping or non-overlapping) patches. One iteration step of the smoothing process consists of solving local mixed problems on each patch one-by-one either in a Jacobi-type or Gauss–Seidel-type manner. This results in an additive or multiplicative Schwarz-type smoother. The technique was successfully used for the Navier–Stokes equations, see \cite{20}. The general construction and the analysis of patch smoothers for mixed problems was discussed in \cite{16}, where a particular patch smoother was proposed for the Stokes problem. An essential feature exploited in the multigrid convergence analysis of the Stokes problem was (in the terminology introduced here) that the adjoint state equation is an elliptic problem in \( y \), where in elliptic optimal control problems the adjoint state equation is typically elliptic in \( p \) but not necessarily in \( y \). Therefore, a straightforward