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Multifractal Formalism for Self-Similar Functions Under the Action of Nonlinear Dynamical Systems

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Abstract. We study functions which are self-similar under the action of some non-linear dynamical systems. We compute the exact pointwise Hölder regularity, then we determine the spectrum of singularities and the Besov "smoothness" index, and finally we prove the multifractal formalism. The main tool in our computation is the wavelet analysis.

1. Introduction

Numerous experimental studies have established the phenomenon of intermittency for the velocity of fully developed turbulence having an irregularly behaved velocity ϑ. The velocity is characterized in this case by local Hölder exponents in a certain interval $[\alpha_{\min}, \alpha_{\max}]$, and each α in this range occurs in a set E^α, which for $0 < \alpha \leq 1$ has the form

$$E^\alpha = \{x \in \mathbf{R}^m \;:\; |\vartheta(x+h,t) - \vartheta(x,t)| \sim |h|^\alpha \text{ as } |h| \mapsto 0\}.$$

The Hausdorff dimension $d(\alpha)$ of E^α is called the spectrum of singularities, and the velocity ϑ is called a *multifractal* function.

In [11], Frisch and Parisi conjectured a formula which relates $d(\alpha)$ to the scaling exponent $\zeta(p)$ of the p-structure functions

$$S_p(h) = \int_{\mathbf{R}^m} |\vartheta(x+h,t) - \vartheta(x,t)|^p \, dx \sim |h|^{\zeta(p)} \qquad \text{as} \quad |h| \mapsto 0.$$

The formula asserts that $d(\alpha)$ is the Legendre transform of $\zeta(p) - m$

$$d(\alpha) = \inf_p(\alpha p - \zeta(p) + m) \tag{1}$$

and this formula is called the *Multifractal Formalism*.

Most examples of multifractal functions follow some self-similarity conditions which state that locally the graph of a multifractal function F is a contraction of the global

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graph modulo an error term g. This means that F satisfies a functional equation of the form

$$F(x) = \sum_{j=1}^{d} \lambda_j F(S_j^{-1}(x)) + g(x) \tag{2}$$

where the S_j are contractions on a bounded set Ω, and the $|\lambda_j|$ are smaller than 1. Our purpose is to determine the spectrum of singularities for such functions and to check when they satisfy the multifractal formalism. Note that some particular cases have been studied (see [2], [8], [9], [16], and [18]) and before being able to state our results we need to review these cases briefly.

The first example is the primitives of some multinomial measures in dimension 1 (i.e., $m = 1$). Let μ be a probability measure supported by $[0, 1]$ and suppose that for any interval J

$$\mu(S_j(J)) = \lambda_j \mu(J) \qquad \forall\, j = 1, \ldots, d,$$

with $\sum_{j=1}^{d} \lambda_j = 1$, and the S_j are linear contractions. Using some ideas of the Holschneider thesis [12], Arneodo, Bacry, and Muzy [2] proved the multifractal formalism for the primitive $F_\mu(x) = \int_0^x d\mu$ in the case where the S_j satisfy the "separated open set condition"

$$S_i([0, 1]) \cap S_j([0, 1]) = \emptyset \qquad \text{if} \quad i \neq j.$$

In [8] and [9], Daubechies and Lagarias proved the multifractal formalism for some refinement functions $\varphi(x) = \sum_{j=0}^{d} \lambda_j \varphi(2x - j)$ (used in the construction of orthonormal wavelet bases) in the case where the coefficients λ_j satisfy the $d - 1$ "sum rules"

$$\sum_{j=0}^{d} \lambda_j (-1)^j j^l = 0 \qquad \text{for} \quad l = 0, \ldots, d - 2.$$

Finally in [16] and [18], Jaffard further extended the multifractal formalism for self-similar functions (in the sense of (2)) in the case where:

- The S_j are contractive similitudes (i.e., the product of an isometry with a homothety of ratio < 1) such that for a bounded open set Ω of $\mathbf{R}^m$

$$S_j(\Omega) \subset \Omega \tag{3}$$

  and

$$S_i(\Omega) \cap S_j(\Omega) = \emptyset \qquad \text{if} \quad i \neq j. \tag{4}$$

- g is a C^k function with all derivatives of order less than k having fast decay.
- There exists $x_0 \in \Omega$ such that $F \notin C^k(x_0)$.

The class of functions of [2] and [8] seems disjoint from the one considered in [18]. In fact, the error term g associated to F_μ (as in (2)) is not Lipschitz and the functions φ do not satisfy the separation condition (4) because they are supported in $[0, d]$. Nevertheless, $F_\mu(x) - x$ is self-similar in the Jaffard sense with a Lipschitz linear function g, and we can use the "sum rules" to find a suitable polynomial $P(x)$ of degree $d - 2$ for which the function $\varphi(x) - P(x)$, restricted to $\Omega =]0, 1[$, satisfies the Jaffard assumptions.