Edge-Pancyclicity and Hamiltonian Connectivity of Twisted Cubes

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Abstract The twisted cube $TQ_n$ is a variant of the hypercube $Q_n$. It has been shown by Chang, Wang and Hsu [Topological properties of twisted cube. Information Science, 113, 147–167 (1999)] that $TQ_n$ contains a cycle of every length from 4 to $2^n$. In this paper, we improve this result by showing that every edge of $TQ_n$ lies on a cycle of every length from 4 to $2^n$ inclusive. We also show that the twisted cube are Hamiltonian connected.

Keywords cycles, twisted cubes, hypercubes, edge-pancyclicity, hamiltonian connectivity

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1 Introduction

The hypercube network has been proved to be one of the most popular interconnection networks. The twisted cubes, proposed first by Hilbers et al. [1], form a class of hypercube variants, and are superior to the hypercube in having about half of the diameter of the hypercube. Various properties of twisted cubes have been investigated in the literature, see, for example, [1–5].

In interconnection networks, the problem of simulating one network by another is modelled as a graph embedding problem. There are several reasons why such an embedding is important [6]. For example, there are a number of efficient algorithms for solving some applications problems and the best communication patterns for their executions. For these algorithms, the existence of certain topological structures guarantees the desired performance. Thus, for such applications, it is desired to provide logically a specific topological structure throughout the execution of the algorithm in the network design.

Among all embedding problems, a cycle embedding problem is one of the most popular problems, that is, find a cycle of given length in graph. A graph $G$ of order $n$ is called $k$-pancyclic [7] if there exists a cycle of every length from $k$ to $n$. The concept of pancyclicity was extended to vertex-pancyclicity by Hobbs [8] and edge-pancyclicity by Alspach and Hare [9], respectively. A graph $G$ is called vertex-$k$-pancyclic (resp. edge-$k$-pancyclic) if for any vertex $u$ (resp. edge $e$), there exists a cycle of every length from $k$ to $n$ containing $u$ (resp. $e$). Clearly, every edge-$k$-pancyclic graph is vertex-$k$-pancyclic. Chang et al. [3] proved that $TQ_n$ is 4-pancyclic since $TQ_n$ contains no cycles of length three. Xu and Ma [10] proved that $TQ_n$
is vertex-4-pancyclic. In this paper, we improve these results by showing that $TQ_n$ is edge-4-pancyclic for $n \geq 3$.

A path is called Hamiltonian if it contains all vertices of $G$. A graph $G$ is said to be Hamiltonian connected if there exists a Hamiltonian path between any pair of vertices of $G$.

In this paper, we prove that the twisted cube is Hamiltonian connected.

The rest of this paper is organized as follows. In Section 2, we give the definition and basic properties of the $n$-dimensional twisted cube $TQ_n$. In Section 3 and Section 4, we discuss the edge-pancyclicity and Hamiltonian connectivity of $TQ_n$, respectively.

## 2 Twisted Cubes

We follow [11] for graph-theoretical terminology and notation not defined here.

The $n$-dimensional twisted cube, first proposed by Hilbers et al. [1], is denoted by $TQ_n$. Its vertex-set consists of all binary strings of length $n$, where $n$ is odd. For a vertex $u = u_{n-1}u_{n-2}\cdots u_1u_0$ in $TQ_n$ and for $0 \leq i \leq n-1$, we define the $i$th parity function $P_i(u) = u_i \oplus u_{i-1} \oplus \cdots \oplus u_0$, where $\oplus$ is the exclusive-or operation. $TQ_n$ can be recursively defined as follows: $TQ_1$ is a complete graph $K_2$ with two vertices 0 and 1. Suppose that $n$ be an odd integer and $n \geq 3$. The vertices of $TQ_n$ can be decomposed into four subsets, $TQ_{n-2}^0, TQ_{n-2}^1$, $TQ_{n-2}^{0,1}$, and $TQ_{n-2}^{1,1}$, where $TQ_{n-2}^{i,j}$ consists of those vertices $u$ with $u_{n-1} = i$ and $u_{n-2} = j$. For each $(i, j) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, the induced subgraph of $TQ_{n-2}^{i,j}$ in $TQ_n$ is isomorphic to $TQ_{n-2}$. Edges which connect these four subtwisted cubes can be described as follows: For a vertex $u = u_{n-1}u_{n-2}\cdots u_1u_0$, if $P_{n-3}(u) = 0$ then it is connected to another vertex $v = u_{n-1}u_{n-2}u_{n-3}\cdots u_3u_0$ or $v = u_{n-1}u_{n-2}u_{n-3}\cdots u_1u_0$; if $P_{n-3}(u) = 1$ then it is connected to $v = u_{n-1}\bar{u}_{n-2}u_{n-3}\cdots u_1u_0$ or $v = \bar{u}_{n-1}u_{n-2}u_{n-3}\cdots u_1u_0$. Figure 1 shows two different but equivalent layouts of $TQ_3$.

\[\text{Figure 1 Equivalent layouts of } TQ_3 \]

Let $G$ be a graph and $K_2$ a complete graph of order two. Use $G \times K_2$ to denote such a graph obtained from two copies of $G$ by adding all edges that join identical vertices in two copies.

The following lemma will be used in the proof of our result.

**Lemma 1** (Huang et al. [5]) The subgraph induced by $TQ_{n-2}^{0,0}\cup TQ_{n-2}^{1,0}$ (resp. $TQ_{n-2}^{0,1}\cup TQ_{n-2}^{1,1}$) is isomorphic to $TQ_{n-2} \times K_2$.

For short, we use $G_0$ and $G_1$ to denote the subgraphs induced by $TQ_{n-2}^{0,0}\cup TQ_{n-2}^{1,0}$ and $TQ_{n-2}^{0,1}\cup TQ_{n-2}^{1,1}$, respectively. We call edges between $G_0$ and $G_1$ critical edges; and call edges