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PROBABILISTIC UNIFORM CONVERGENCE SPACES REDEFINED

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Abstract. We develop a theory of probabilistic uniform convergence spaces based on Tardiff's neighbourhood systems for probabilistic metric spaces. We show that the resulting category is topological and Cartesian closed. A subcategory is identified that is isomorphic to the category of probabilistic metric spaces.

1. Introduction

Probabilistic uniform convergence spaces were first defined by Nusser [6]. They form a generalization of both uniform convergence spaces as defined by Cook and Fischer [2] (and improved by Wyler [13]) and probabilistic uniform spaces as defined by Florescu [3]. A probabilistic uniform convergence structure can loosely be described as a "tower" of uniform convergence structures, indexed by the unit interval $[0,1]$. In this paper, we generalize Nusser's definition by replacing the "index set" $[0,1]$ by the set Δ^+ of distance distribution functions [10]. A similar idea was used by Tardiff [12] to generate a family of neighbourhood structures, and thus a family of topologies, for a probabilistic metric space. Although the use of a fixed distance distribution function, a so-called *profile function* $\varphi \in \Delta^*$, allows a probabilistic interpretation (see e.g. [4,10]), Tardiff does not attach such an interpretation to his neighbourhood systems. Likewise, we also do not see such an interpretation in our case but use the new index set solely as a technical tool and keep the name "probabilistic uniform convergence space" because of the similarity of our spaces with the ones of Nusser and the relation to probabilistic metric spaces.

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Our generalization solves two problems. Firstly, just like metric spaces carry in a natural way a uniform structure (and hence also a uniform convergence structure), we can show, using an idea of Tardiff [12], that each probabilistic metric space carries a natural probabilistic uniform structure. Secondly, we can identify a subcategory of the category of probabilistic uniform convergence spaces that is isomorphic to the category of probabilistic metric spaces. In this sense, we can characterize probabilistic metric spaces entirely by their probabilistic uniform (convergence) structure.

2. Preliminaries

For an ordered set $(A, \leq)$ we denote, in case of existence, by $\bigwedge_{i\in I} \alpha_i$ the infimum and by $\bigvee_{i\in I} \alpha_i$ the supremum of $\{\alpha_i : i \in I\} \subseteqq A$. In case of a two-point set $\{\alpha, \beta\}$ we write $\alpha \wedge \beta$ and $\alpha \vee \beta$, respectively.

For a set S, we denote its power set by $P(S)$ and the set of all filters $\mathbb{F}$, $\mathbb{G}$, ... on S by $\mathbb{F}(S)$. The set $\mathbb{F}(S)$ is ordered by set inclusion and maximal elements of $\mathbb{F}(S)$ in this order are called *ultrafilters*. In particular, for each $p \in S$, the point filter $[p] = \{A \subseteqq S : p \in A\} \in \mathbb{F}(S)$ is an ultrafilter. For $\mathbb{F} \in \mathbb{F}(S)$ and $\mathbb{G} \in \mathbb{F}(T)$ we denote $\mathbb{F} \times \mathbb{G}$ the filter on $S \times T$ generated by the sets $F \times G$ where $F \in \mathbb{F}$ and $G \in \mathbb{G}$. For $\Phi, \Psi \in \mathbb{F}(S \times S)$ we define Φ^{-1} to be the filter generated by the sets $\phi^{-1} = \{(p, q) \in S \times S : (q, p) \in \phi\}$ and if for all $\phi \in \Phi$ and $\psi \in \Psi$ the sets $\phi \circ \psi = \{(p, q) \in S \times S : \exists r \in S$ such that $(p, r) \in \phi, (r, q) \in \psi\}$ are non-empty, we denote the filter generated by these sets by $\Phi \circ \Psi$.

We assume some familiarity with category theory and refer to the textbooks [1] and [7] for more details. A *construct* is a category $\mathcal{C}$ whose objects are structured sets (S, ξ) and morphisms are suitable mappings between the underlying sets. A construct is called *topological* if it allows *initial constructions*, i.e. if for every source $(f_i : S \longrightarrow (S_i, \xi_i))_{i\in I}$ there is a unique structure ξ on S, such that a mapping $g : (T, \eta) \longrightarrow (S, \xi)$ is a morphism if and only if for each $i \in I$ the composition $f_i \circ g : (T, \eta) \longrightarrow (S_i, \xi_i)$ is a morphism.

A topological construct is called *Cartesian closed* if for each pair of objects (S, ξ), (T, η) there is a function space structure on the set $C(S, T)$ of morphisms from S to T such that the *evaluation mapping* $\mathrm{ev} : C(S, T) \times S \longrightarrow T$, $(f, s) \longmapsto f(s)$ is a morphism and that for each object (Z, ζ) and each morphism $f : S \times Z \longrightarrow T$ the mapping $f^* : Z \longrightarrow C(S, T)$ defined by $f^*(z)(x) = f(x, z)$ is a morphism [7].

A function $\varphi : [0, \infty] \longrightarrow [0, 1]$, which is non-decreasing, left-continuous on $(0, \infty)$ and satisfies $\varphi(0) = 0$ and $\varphi(\infty) = 1$ is called a *distance distribution function* [10]. The set of all distance distribution functions is denoted by Δ^+. For example, for each $0 \leq a < \infty$ the functions

$$\varepsilon_a(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq a \\ 1 & \text{if } a < x \leq \infty \end{cases} \qquad \text{and} \qquad \varepsilon_\infty(x) = \begin{cases} 0 & \text{if } 0 \leq x < \infty \\ 1 & \text{if } x = \infty \end{cases}$$