DETERMINATE SYSTEMS

Estimation of a Local Lipschitz Constant of the $B_q$-Entropy

Yu. S. Popkov and M. V. Rublev

Institute for Systems Analysis, Russian Academy of Sciences, Moscow, Russia

Received August 10, 2004

Abstract—A method for estimating a Lipschitz constant of the entropy operator of the Boltzmann type is suggested. Examples of the use of the obtained estimates in problems for restoring images by projections are given.

1. INTRODUCTION

A considerable number of the applied problems of macrosystem modeling rests on the use of various types of entropy operators [1–3]. The common structure of an entropy operator has the form

$$u(v) = \arg \max_u (H(u, v)|u \in D(v)), \quad u \in R_m^+, \quad v \in Q \subset R_s^+,$$

where $H(u, v)$ is the entropy function or entropy functional and $v$ is the vector parameter. In the problems of modeling of equilibrium states of macrosystems, components of the vector $v$ have the sense of external parameters that exert an action on a solution of problem (1.1) [1]. If the operator $u(v)$ is used for developing the dynamic models of macrosystems, then $v$ is the vector function of time [4].

Here, we will deal with the so-called $B_q$-entropy operators

$$u(q) = \arg \max_u (H(u)|u \in D(q)), \quad u \in R_m^+, \quad q \in Q \subset R_r^+,$$

where $H(u)$ is the Boltzmann entropy [1], $Q$ is the compact set, and the set

$$D(q) = \left\{ u : \sum_{i=1}^{m} t_{ki}u_i = q_k, \quad k = 1, \ldots, r \right\},$$

$$t_{ki} \geq 0, \quad k = 1, \ldots, r; \quad i = 1, \ldots, m.$$

This work is meant for the development of a method for estimating a local Lipschitz constant (on a compact set) for the entropy operators of the given class.

2. STATEMENT OF THE PROBLEM

We will consider operator (1.2) and convert it to the form that is more convenient for the investigation. We introduce the designation

$$\tau_i = \sum_{k=1}^{r} t_{ki}, \quad i = 1, \ldots, m.$$
Let us prescribe the variables
\[ x_i = \tau_i u_i, \quad t_{ki} = \frac{t_{ki}}{\tau_i} \geq 0. \] (2.2)

We will examine the \( B_q \)-entropy operator of the form
\[ x(q) = \arg \max_x (H(x)|Tx = q), \quad q \in Q \subset R^n_+, \quad x \in R^n_+, \] (2.3)
where \( H(x) \) is the generalized Boltzmann information entropy
\[ H(x) = -\sum_{i=1}^{m} x_i \ln \frac{x_i}{a_i e^a}, \quad a_i \geq 0, \quad i = 1, \ldots, m; \] (2.4)
and \( T \) is the \((r \times m)\) matrix of the complete rank \( r \) with the nonnegative elements (2.2) and the normalized elements
\[ \sum_{k=1}^{r} t_{ki} = 1, \quad i = 1, \ldots, m; \quad k = 1, \ldots, r. \] (2.5)

We will assume that for the matrix \( TT^* \) (the sign \( * \) denotes the operation of transposition), the following conditions (of the dominating diagonal) are fulfilled:
\[ \sum_{i=1}^{m} t^2_{ki} - \sum_{j \neq k}^{r} \sum_{i=1}^{m} t_{ki} t_{ji} \geq \delta > 0, \quad k = 1, \ldots, r. \] (2.6)

According to (2.2), the following relation between \( B_q \) (2.3) and the entropy operators \( \tilde{B}_q \) (1.2) exists:
\[ u(q) = \tau^{-1} \otimes x(q), \] (2.7)
where \( \otimes \) implies the coordinate-wise multiplication.

We will consider an auxiliary matrix \( C \) with the nonnegative elements
\[ c_{kj} = \sum_{i=1}^{m} a_i t_{ki} t_{ji} \geq 0, \quad k, j = 1, \ldots, r, \] (2.8)
whose determinant is
\[ \det(C) \neq 0. \] (2.9)

Let us denote by \( \det(C^k) \) the determinant of the matrix \( C^k \) formed by changing the \( k \)th column for the vector \( q \) (see (2.3)). This determinant can be represented in the form
\[ \det(C^k) = \sum_{i=1}^{r} A_{ki} q_i, \quad A_{ki} = (-1)^{k+i} M_{ki}, \] (2.10)
where \( M_{ki} \) is the \((k, i)\)-minor of the determinant of the matrix \( C \) (2.8).