**H∞ Filtering for Discrete-Time Systems with Time-Varying Delay**

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**Abstract:** The problem of $H∞$ filtering for discrete-time systems with time-varying delay in measurement is investigated in this paper. First, under the assumption that the time-varying delay is of a known upper bound, the delayed measurement is re-described as the one with multiple state delays. Then the proposed $H∞$ filtering problem is transformed into one for systems with multiple measurement channels that contain the same state information as the original measurement and each channel has a single constant delay. Finally, based on the reorganized innovation analysis approach in Krein space, a necessary and sufficient condition for the existence of an $H∞$ filter which guarantees a prescribed attenuation level is derived. The solution to the $H∞$ filtering is given in terms of the solutions to Riccati and matrix difference equations.

**Keywords:** $H∞$ filtering, discrete-time systems, reorganized innovation analysis, Riccati difference equations, time-varying delay.

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**1. INTRODUCTION**

Linear estimation of dynamic systems has many practical applications in the filed of science and engineering and has attracted much attention during the last decades. The celebrated Kalman filtering has been one of the most common and successful state estimation technique in a variety of applications. In contrast with the traditional Kalman filtering, the $H∞$ filtering approach does not require the exact knowledge of the statistics of the external noise sources, which can be arbitrary signals with bounded energy. $H∞$ filtering is concerned with the design of estimators which ensure a bound on the $L∞$ induced gain from disturbance signals to estimation errors. Various approaches have been developed in the past decades to deal with the $H∞$ filtering problems in various setting such as deterministic systems with uncertainties or delays as well as various stochastic systems. In earlier works, several algorithms without considering time delay were derived by a variety of methods in both continuous- and discrete-time cases. Most solutions to the issue were given by using the duality with the known solution of the $H∞$ control [1-4], others were given by using the game-theoretic techniques and dynamic programming [5,6], or frequency domain techniques based on polynomial factorization [7,8].

The study of the $H∞$ filtering problem for systems with delays has gained growing interest in recent years. In [9], $H∞$ filtering for systems with a time delay in the measurement was discussed based on a Riccati equation approach. The $H∞$ observer design for systems with state delays was considered in [10], where a sufficient condition based an algebraic Riccati equation was derived. Robust $H∞$ filtering of uncertain systems with state delays has been fully considered in [11-15] where both delay-independent and delay-dependent sufficient conditions have been derived. The prevailing methods are based on bounded real lemma in terms of Riccati algebraic equations or linear matrix inequalities.

However, it should be noted that the $H∞$ filtering problems in the above mentioned references are consider with constant time delays. Very recently, the interest on the estimation and control for systems with time-varying delay and data loss is growing [16,17]. It is due to that a growing number of applications demands remote control of plants over unreliable networks, such as wireless sensor networks. In [18], $H∞$ filtering for continuous-time systems with time-varying interval delay is considered by using the free-weighting matrix method, while the discrete-time counterpart in [19]. $H∞$ filtering for systems with time-varying distributed delays was investigated based on LMIs in [20]. Robust $H∞$ filtering of uncertain continuous-time systems with time-varying delays has been considered in terms of LMIs in [21-24] where two cases of time-varying delays were considered, one is the time-varying delay being continuous uniformly bounded while the other is the time-varying delay being differentiable uniformly bounded with delay-derivative bounded by a constant. In [25], $H∞$ filtering problem for Markovian jump linear systems with norm-bounded parameter uncertainties and time-varying delays was
The time-varying delays considered in the aforementioned works are all in state. However, in many cases, we have to estimate system states with time-varying delayed outputs. For example, in wireless sensor networks, as well as networked control systems containing a large number of sensors, controllers and actuators, the sensing information of the system state have to be transferred via an unreliable network, where the data may be out-of-order to the estimators. In such situations, state delays will appear in the measurement equation of the state-space model. The state estimation problems for discrete- and continuous-time systems with time-varying delay in measurement have been considered under $H_2$ performance in [26] and [27], respectively. Under $H_\infty$ performance, the estimation problem was considered in [28] based on LMIs.

In this paper, we investigate the $H_\infty$ filtering problem for discrete-time systems with time-varying delay in measurements by combining the reorganized innovation analysis approach and the Krein space estimation theory (usually, this method is termed as the reorganized innovation analysis approach in Krein space). The reorganized innovation analysis approach, which is based on projection and a reorganized innovation sequence for deriving the filter, first appeared in [29] and [30] to deal with $H_2$ estimation problems with time delay. The Krein space approach, which provides the unified geometric formulation, allows solving seemingly different estimation problems with different deterministic and stochastic criteria by formulating them in certain indefinite inner products spaces, first appeared in [31] and [32] to deal with the $H_\infty$ estimation problems without time delays. The reorganized innovation analysis approach in Krein space is first used to deal with the $H_\infty$ filtering problem for continuous-time systems with constant delay and the $H_\infty$ fix-lag smoothing in [33] and [34], respectively, where the innovation matrices appeared in the Riccati equations and the filter gain formulas are nonsingular, the gain formulas are always solvable and the solutions of the filter gains are unique. In this paper, we use this method to tackle a more difficult $H_\infty$ estimation problem where time-varying delay is considered. Under the assumption that time-varying delay is bounded, the problem is first transformed, by defining a binary variable with value 0 or 1, into the one with multiple channels, each of which has a constant delay. However, due to the transformation of the system, the innovation matrices appeared in the Riccati equations and the filter gain formulas may be singular. The solvability of the gain formula and the uniqueness of the solution to the Riccati equation will be discussed. The solution to $H_\infty$ filtering with time-varying delay is given in terms of solutions to Riccati and matrix difference equations.

The notation used in the paper is fairly standard. $A'$ stands for the transposition of a matrix $A$, $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space, $\mathbb{R}^{n\times r}$ is the set of all $n \times r$ real matrices, and the notation $B>0$ means that a matrix $B$ is symmetric and positive definite. diag{$\cdots$} denotes a block-diagonal matrix, whereas col{$\cdots$} stands for a column vector. $I_p$ denotes the $p \times p$ identity matrix. Additionally, for the convenience of interpretation, we use $I$ and $0$ in boldface to denote the identity and the zero matrices of appropriate dimensions, respectively.

### 2. PROBLEM FORMULATION

Consider the following linear discrete-time system for $H_\infty$ filtering problem:

\[
x(t+1) = \Phi(t)x(t) + \Gamma(t)e(t),
\]

\[
y(t) = H(t)x(t-h(t)) + v(t),
\]

\[
z(t) = L(t)x(t),
\]

where $x(t) \in \mathbb{R}^n$ is the system state vector with initial state $x_0$; $y(t) \in \mathbb{R}^p$ is the delayed measurement vector; $e(t) \in \mathbb{R}^r$ and $v(t) \in \mathbb{R}^p$ are the process noise and exogenous disturbance signal, which are assumed to be arbitrary in $L_2^2(0, N)$, where $N > 0$ is the time-horizon of the $H_\infty$ filtering problem under consideration; $z(t)$ is a linear combination of state variables to be estimated; $\Phi(t) \in \mathbb{R}^{n\times n}$, $\Gamma(t) \in \mathbb{R}^{n\times r}$, $H(t) \in \mathbb{R}^{p\times n}$ and $L(t) \in \mathbb{R}^{p\times r}$ are known bounded time-varying matrices.

**Assumption 1:** $h(t)$ in (2) is the positive integer time-varying delay term satisfying $0 \leq h(t) \leq m$, where $m$ is a constant number. Moreover, it is always assumed that $h(t) \leq t$ for any $t \in \{0, \ldots, N\}$ throughout this paper.

Now we define a variable denoted as $\beta_{i,j}$ to model the observation arrival process,

\[
\beta_{i,j} \triangleq \begin{cases} 1, & \text{the observation delay } h(t) = i(\leq m); \\ 0, & \text{otherwise}. \end{cases}
\]

At an arbitrary discrete time $t$, it is assumed that $h(t)$ can only choose one value from the finite set, then we get the following property of $\beta_{i,j}$,

\[
\beta_{i,j} \times \beta_{i',j'} = \begin{cases} 0, & \text{if } i \neq j; \\ 1, & \text{if } i = j. \end{cases}
\]

Furthermore, the observation data is time-stamped to estimator, thus the coefficient $\beta_{i,j}$ is known for any $i = 0, 1, \ldots, m$. Then the observation equation (2) can be reformulated equivalently as:

\[
y(t) = \begin{cases} \sum_{i=0}^{t} \beta_{i,j} H(t)x(t-i) + v(t), & t < m; \\ \sum_{i=0}^{m} \beta_{i,j} H(t)x(t-i) + v(t), & t \geq m. \end{cases}
\]