Robust $H_\infty$ Static Output Feedback Stabilization of T-S Fuzzy Systems Subject to Actuator Saturation

Dounia Saifia, Mohammed Chadli*, Salim Labiod, and Thierry Marie Guerra

Abstract: This paper proposes a method for designing robust $H_\infty$ static output feedback stabilization of Takagi-Sugeno (T-S) fuzzy systems under actuator saturation. In this paper, the input saturation is represented by a polytopic model and the modeling error is assumed a norm-bounded uncertainty. A set invariance condition for robust $H_\infty$ static output feedback system under actuator saturation is first established. Then, the estimation of the largest domain of attraction for the system is formulated and solved as a Linear Matrix Inequality (LMI) optimization problem. Two examples are used to demonstrate the effectiveness of the proposed design method.

Keywords: Actuator saturation, $H_\infty$ static output-feedback stabilization, Takagi-Sugeno (T-S) fuzzy system, uncertainties.

1. INTRODUCTION

The Takagi-Sugeno fuzzy model has been successfully used to approximate nonlinear systems by interpolation of numerous local linear models in terms of IF-THEN fuzzy rules [1]. This approach has been extensively used for stability analysis and control of nonlinear systems [2-9]. However, since state variables are not always fully measurable for most industrial plants, many researchers paid more attention to output feedback control and particularly for dynamic output feedback [10,11]. The drawback of this kind of control is the increasing dimension of the closed-loop system. Alternatively, static output feedback control is proposed in numerous works (see for example [12-17]).

On the other hand, in practice, all actuators have physical limitations which would lead to performance degradation and often make the stable closed-loop system unstable [3,10,18-22]. Thus, the implementation of control laws designed without taking into account the saturation effect may have undesirable consequences on the system behavior. This problem has been receiving increasing attention for control of both linear [11,18-20,23-26] and nonlinear systems [3,10,22,27,28]. Generally, the saturation limits are avoided by designing low gain control laws and considering a bounded set of initial system states [3,21]. However, this method often gives low levels of performance [3]. Alternatively, the problem is dealt with by estimating the domain of attraction of the closed-loop system in the presence of actuator saturation [3,8,10,19]. In most cases, the saturation nonlinearity term is transformed into dead zone nonlinearity [10,18,24-26] or it is represented by a polytopic model [3,11,20,22]. Based on the last representation and according to a quadratic Lyapunov function, stability conditions for nonlinear systems under actuator saturation are extracted, formulated and solved as a linear matrix inequality (LMI) optimization problem [3,10,21,22].

The problem of robust $H_\infty$ static output feedback stabilization with uncertainties and actuators saturation was performed in the case of linear systems [29], where the sector condition is used to derive the stability conditions and to formulate them as a bilinear optimization problem. In the case of nonlinear systems, many results on $H_\infty$ static output-feedback stabilization exist in the literature [17,30]. However, to our knowledge, there is no results for $H_\infty$ static output-feedback stabilization for nonlinear systems with actuator saturation and nonlinear state-dependent outputs.

This paper deals with the robust $H_\infty$ static output feedback stabilization of T-S fuzzy systems subject to actuator saturation, parametric uncertainties and external disturbances. The saturation effect is represented by a polytopic model. The problem of estimating the largest domain of attraction for the robust $H_\infty$ static output feedback system with actuator saturation is formulated and solved as a LMI optimization problem. This paper is organized as follows. Section 2 gives mathematical description of uncertain T-S models with static output feedback in the presence of saturation.
Section 3 gives robust H\(\infty\) static output feedback stabilization conditions in term of LMI optimization problem. In Section 4, the estimation of the largest domain of attraction for the close-loop system is formulated and solved as LMI terms. Finally, in Section 5, a numerical example and a permanent magnetic synchronous motor (PMSM) are given to show the effectiveness of the proposed method.

**Notation:** \(I\) denotes the set \([1, 2, \ldots, r]\), \(\mathbb{R}^{m \times n}\) the set of all \(n \times m\) real matrices. \(M > (\geq, <, \leq)\) is used to denote a symmetric positive definite (positive semi-definite, negative definite, negative semi-definite, respectively) matrix. \((*)\) denotes the symmetric bloc matrix, \(X + (*)\) denotes \(X + X^T\).

## 2. PRELIMINARIES

### 2.1. Takagi-Sugeno fuzzy systems

A T-S fuzzy system with a norm-bounded uncertainty is represented by the following fuzzy rules [10,30-32]:

\[ R_i: \text{If } \xi_i(t) \text{ is about } M_{ij}, \text{ and } \cdots \text{and } \xi_i(t) \text{ is about } M_{ij}, \text{ then } \]

\[ \begin{align*}
\dot{x}(t) &= \tilde{A}_i x(t) + \tilde{B}_i w(t) + \tilde{B}_2 \sigma(t) \\
\hat{z}(t) &= \tilde{C}_i x(t) + \tilde{D}_i w(t) + \tilde{D}_2 \sigma(t) \\
y(t) &= \tilde{C}_2 x(t),
\end{align*} \tag{1a} \]

For \(i \in I\), in which \(M_{ij}\) is the fuzzy set of \(\xi_i\) in rule \(R_i\), \(r\) is the number of IF-THEN fuzzy rules and \(\xi_i(t)\) are the decision variables assumed measurable, \(x(t) \in \mathbb{R}^n\) is the system state vector, \(\sigma(t) \in \mathbb{R}^m\) is the saturate control input, \(y(t) \in \mathbb{R}^p\) is the measurable output, \(z(t) \in \mathbb{R}^q\) is the controlled output variable, \(w(t) \in \mathbb{I}\) is the disturbance variable with \(\mathbb{I}\) = \(\{w \in \mathbb{R}^m : \|w\|_2 \leq \bar{w}, \bar{w} > 0\}\).

The global dynamic is inferred as:

\[ \begin{align*}
\dot{x}(t) &= \sum_{i=1}^{r} \mu_i(\xi_i(t)) \left( \tilde{A}_i x(t) + \tilde{B}_i w(t) + \tilde{B}_2 \sigma(t) \right) \\
\hat{z}(t) &= \sum_{i=1}^{r} \mu_i(\xi_i(t)) \left( \tilde{C}_i x(t) + \tilde{D}_i w(t) + \tilde{D}_2 \sigma(t) \right) \\
y(t) &= \sum_{i=1}^{r} \mu_i(\xi_i(t)) \tilde{C}_2 x(t),
\end{align*} \tag{1a} \]

where

\[ \mu_i(\xi_i(t)) = \frac{\prod_{j=1}^{q} M_{ji}(\xi_j(t))}{\sum_{j=1}^{r} \prod_{j=1}^{q} M_{ji}(\xi_j(t))} \tag{1b} \]

\(M_{ji}(\xi_j(t))\) is the grade of membership of \(\xi_j(t)\) in \(M_{ji}\).

The normalized activation function \(\mu_i(\xi_i(t))\) in relation with the \(i^{th}\) sub-model is such that: \(\sum_{i=1}^{r} \mu_i(\xi_i(t)) = 1\) where \(0 \leq \mu_i(\xi_i(t)) \leq 1\) \(\forall i \in I_i\).

The considered parametric uncertainties are as follows:

\[ \begin{align*}
\dot{\hat{A}}_i &= A_i + \Delta A_i, \quad \dot{\hat{B}}_i = B_i + \Delta B_i, \quad \dot{\hat{C}}_i = C_i + \Delta C_i, \\
\dot{\hat{D}}_i &= D_i + \Delta D_i, \quad \dot{\hat{D}}_2 = D_2 + \Delta D_2, \quad \dot{\hat{C}}_2 = C_2, \\
\Delta A &= \Delta B_1, \quad \Delta B_2 = M \Theta(N_A, N_B_1, N_B_2), \\
\Delta C_1 &= \Delta D_1, \quad \Delta D_2 = M \Theta(N_D_1, N_D_2),
\end{align*} \tag{1c} \]

where \(M, M_1, N_A, N_B_1, N_B_2, N_D_1, N_D_2\) and \(N_2\) are constant matrices with compatible dimensions and \(\Theta^T(t) \Theta(t) \leq I\).

### 2.2. Saturation function

The control input is subject to actuator saturation:

\[ \sigma(t) = \text{sat}(u(t), \bar{u}), \tag{2} \]

\[ \text{sat}(u(t), \bar{u}) = [s_1, s_2, \ldots, s_m]^T, \tag{3} \]

where \(\bar{u} \in \mathbb{R}^m\) denotes the saturation level and \(u(t) \in \mathbb{R}^n\) is the control input, \(\bar{u}_i\) and \(u_i\) denote the \(i^{th}\) element of \(\bar{u}\) and \(u\), respectively.

For modeling the saturation effect, the polytopic representation method proposed in [3,22] will be used.

**Lemma 1:** Let \(E\) be the set of \(m \times n\) diagonal matrices whose diagonal elements are 1 or 0. Suppose that \(|v_i| \leq \bar{u}_i\) for all \(i \in I_m\) where \(v_i\) and \(u_i\) denote the \(i^{th}\) element of \(v \in \mathbb{R}^n\) and \(u \in \mathbb{R}^n\), respectively. If \(x \in \bigcap_{j=1}^{r} \mathcal{J}(H_j)\) for \(x \in \mathbb{R}^n\), then:

\[ \text{sat}(u, \bar{u}) = \sum_{j=1}^{m} \alpha_j E_j u + E_j^{-1} v, \quad v = \sum_{j=1}^{r} \mu_j(\xi) H_j x \quad \text{satisfy} \quad \sum_{j=1}^{m} \alpha_j = 1, \quad 0 \leq \alpha_j \leq 1, \tag{4} \]

\[ \mathcal{J}(H_j) = \left\{ x \in \mathbb{R}^n \mid h_j^T x \leq \bar{u}_j \right\}, \tag{5} \]

where \(E_j\) denotes all elements of \(E\), \(E_j^{-1} = I - E_j\), \(H_j\) is \(m \times n\) matrix and \(h_j^T\) is the \(i^{th}\) row of \(H_j\).

### 2.3. Quadratic Lyapunov stability

For a symmetric positive matrix \(P \in \mathbb{R}^{m \times n}\), we define a Lyapunov function as follows:

\[ V(x) = x^T P x(t), \quad P > 0. \tag{6} \]

For a constant \(\rho > 0\), define an ellipsoid:

\[ \mathcal{E}(P, \rho) = \left\{ x \in \mathbb{R}^n \mid x^T P x \leq \rho \right\}. \tag{7} \]

An ellipsoid \(\mathcal{E}(P, \rho)\) is said to be contractively invariant set if \(\dot{V}(x) < 0, \forall x \in \mathcal{E}(P, \rho)\). Therefore, if an