Differ Equ Dyn Syst (April 2012) 20(2):139–148
DOI 10.1007/s12591-012-0110-1

ORIGINAL RESEARCH

Integro-Differential Equations of Fractional Order

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Published online: 5 April 2012
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Abstract In this paper, the authors present some results concerning the existence and uniqueness of solutions of an integro-differential equation of fractional order by using Banach's contraction principle, Schauder's fixed point theorem, and the nonlinear alternative of Leray–Schauder type.

Keywords Integro-differential equation · Left-sided mixed Riemann–Liouville integral of fractional order · Caputo fractional-order derivative, solution · Banach's contraction principle · Schauder's fixed point theorem · Nonlinear alternative of Leray–Schauder type

Introduction

Fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus itself starting from some speculations of Leibniz (1697) and Euler (1730) and progressing to the present day (see [15]). Fractional integral equations have recently been applied in various areas of engineering, science, finance, applied mathematics, bio-engineering, and elsewhere. However, many researchers remain unaware of this field. There has been a significant development in ordinary and partial fractional differential equations in recent years; see, for example, the

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monographs of Kilbas et al. [7] and Podlubny [13], the papers of Abbas and Benchohra [2] and Pachpatte [11,12], as well as the references contained therein.

In [10], Mureşan proved some results on the existence, uniqueness, and data dependence as well as comparison theorems by applying results of Picard on weakly Picard operator theory (see [14]) to functional integral equations of the form

$$x(t) = \alpha + f\left(x, \int_0^{g(t)} x(s)ds, x(h(t))\right), \quad t \in [0, T], \tag{1}$$

where $T > 0$, $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, and $g, h : [0, T] \to [0, T]$.

In this paper we improve some of the above results for integro-differential equations of fractional order of the form

$$^cD_\theta^r u(x, y) = f(x, y, I_\theta^r u(x, y), u(x, y)), \quad (x, y) \in J := [0, a] \times [0, b], \tag{2}$$

$$\begin{cases} u(x, 0) = \varphi(x), & x \in [0, a], \\ u(0, y) = \psi(y), & y \in [0, b], \\ \varphi(0) = \psi(0), \end{cases} \tag{3}$$

where $\theta = (0, 0), r = (r_1, r_2), r_1, r_2 \in (0, \infty)$, I_θ^r is the left-sided mixed Riemann-Liouville integral of order r, $^cD_\theta^r$ is the fractional Caputo derivative of order r, $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a given continuous function, and $\varphi : [0, a] \to \mathbb{R}^n$ and $\psi : [0, b] \to \mathbb{R}^n$ are absolutely continuous functions. Our investigations on the existence and uniqueness of solutions to our problem are conducted with an application of Banach's contraction principle, Schauder's fixed point theorem, and the nonlinear alternative of Leray-Schauder type.

Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that are used throughout the paper. We let $C(J)$ denote the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm

$$\|w\|_\infty = \sup_{(x,y)\in J} \|w(x, y)\|,$$

where $\|\cdot\|$ denotes a suitable complete norm on $\mathbb{R}^n$. Let E be the space of functions $w \in C(J)$ that satisfy the condition:

$$\text{there exists } M \geq 0 \text{ such that } \|w(x, y)\| \leq Me^{\lambda(x+y)} \text{ for } (x, y) \in J, \tag{4}$$

where λ is a positive constant. In the space E we define the norm

$$\|w\|_E = \sup_{(x,y)\in J} \{\|w(x, y)\|e^{-\lambda(x+y)}\}.$$

According to [9], $(E, \|\cdot\|_E)$ is a Banach space. The above definition of $\|\cdot\|_E$ is a variation of Bielecki's norm [3]. We note that the condition (4) implies that

$$\|w\|_E \leq M. \tag{5}$$

As is usual, $AC(J)$ denotes the space of absolutely continuous functions from J into $\mathbb{R}^n$ and $L^1(J)$ is the space of Lebesgue-integrable functions $w : J \to \mathbb{R}^n$ with the norm

$$\|w\|_{L^1} = \int_0^a \int_0^b \|w(x, y)\|dydx.$$