CAN CONSTRUCTIVE MATHEMATICS BE APPLIED IN PHYSICS?

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ABSTRACT. The nature of modern constructive mathematics, and its applications, actual and potential, to classical and quantum physics, are discussed.

KEY WORDS: constructive mathematics, quantum mechanics, recursive, unbounded operator

1. INTRODUCTION

In this article I would like first to explain what some practitioners of ‘constructive mathematics’ mean by that phrase, and then to examine certain aspects of classical and quantum physics from a constructive point of view. In particular, having discussed constructive mathematics in general in Section 1, I shall deal, in the succeeding sections, with the constructive significance of the work of Pour-El and Richards, the reality of a constructive theory of unbounded operators, and constructive mathematics as a medium for the foundations of quantum physics.

There are several substantial references for constructive mathematics to which the reader can refer for more details on the subject; among these are [1, 2, 3, 9, 22, 27].

2. CONSTRUCTIVE MATHEMATICS

What is constructive mathematics? According to the pioneers of the subject, such as Brouwer [11], Markov [19, 20], and Bishop [2], it is mathematics carried out under the requirement that ‘existence’ must be strictly interpreted as ‘computability’ (in some sense that depends on the particular outlook of the individual pioneer). If you adopt this view, then, almost inevitably, you find yourself using a different logic from the usual classical one.

This new logic, known as intuitionistic logic, was originally abstracted from constructive mathematical practice. For Brouwer, at least, mathematics had priority over logic; the latter was derived from a close in-
spection of the principles needed, and in some cases rejected, when one works constructively. On the other hand, Brouwer’s introspection led him to formulate a number of principles of his ‘intuitionism’ that led to results apparently in conflict with aspects of classical mathematics.

This gave rise to the traditional view of constructive mathematics as an approach, based on hard-line philosophical views about existence, that leads to the replacement of basic, and frequently perceived as essential, rules of logic by eccentric principles that may lead to ‘theorems’ flatly contradicting well-known results of classical mathematics. Although this view represents a gross misunderstanding of Brouwer’s ideas, and an equally gross undervaluation of his contribution to the clarification of distinctions between the classical and constructive approaches, in light of the intractability of Brouwer and some other leading exponents of constructivism, it is not surprising that classical mathematicians, led by Hilbert and believing that their life’s work was under attack, have often reacted to constructive mathematics with vigorous opposition.

There is, however, a polemic-free, modern view of constructive mathematics. This view, propounded by Fred Richman [24] and which I share, is based on our thirty years of investigating constructive mathematics in the style of Errett Bishop [2]. It seems clear to us that, in practice,

constructive mathematics is none other than
mathematics carried out with intuitionistic logic

For us, the distinctive feature of constructive mathematics is its methodology, rather than its content. In other words, we can work constructively – that is, using intuitionistic logic – with any objects of mathematics, not just those of some special ‘constructive’ type (whatever that may mean).

But why would anyone choose to work in this way? Because mathematics developed with intuitionistic logic has more interpretations/models than its classical counterpart. In particular, every theorem proved with intuitionistic logic can be modelled within

- recursive function theory (one form of computability theory with a very precise notion of algorithm);
- Brouwer’s intuitionistic mathematics;
- any formal system for computable mathematics that I know of, such as Weihrauch’s TTE [26, 28, 29] or Martin-Löf’s Theory of Types [21]; and
- classical (that is, traditional) mathematics.

So, at the cost of some logical principles, we gain more models. (The computational models are, of course, the ones that we are most interested