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Multi-Fractal Formalism for Quasi-Self-Similar Functions

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The study of multi-fractal functions has proved important in several domains of physics. Some physical phenomena such as fully developed turbulence or diffusion limited aggregates seem to exhibit some sort of self-similarity. The validity of the multi-fractal formalism has been proved to be valid for self-similar functions. But, multi-fractals encountered in physics or image processing are not exactly self-similar. For this reason, we extend the validity of the multi-fractal formalism for a class of some non-self-similar functions. Our functions are written as the superposition of "similar" structures at different scales, reminiscent of some possible modelization of turbulence or cascade models. Their expressions look also like wavelet decompositions. For the computation of their spectrum of singularities, it is unknown how to construct Gibbs measures. However, it suffices to use measures constructed according the Frostman's method. Besides, we compute the box dimension of the graphs.

KEY WORDS: Multi-fractal formalism; wavelets; turbulence; cascade models; Gibbs measures; non-self-similar functions; Frostman's method; box dimension.

1. INTRODUCTION

A bounded function $F: \mathbb{R}^m \to \mathbb{C}$ is $C^\alpha(x_0)$ for $\alpha > 0$ if there exists a polynomial P of degree at most $[\alpha]$ and a constant C such that, if $|x-x_0| \leqslant 1$,

$$|F(x)-P(x-x_0)| \leqslant C\,|x-x_0|^\alpha \tag{1}$$

A function F belongs to $C^\alpha(\mathbb{R}^m)$ if (1) holds for any x and x_0 in $\mathbb{R}^m$ with a uniform constant C.

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In order to characterize both the regularity and the irregularity of F at x_0, we define the Hölder exponent $\alpha_F(x_0)$ of F at x_0 as the supremum of all values of α such that F is $C^\alpha(x_0)$. If F is n times continuously differentiable at the point x_0 then one can use for the polynomial $P(x-x_0)$ the order n Taylor series of F at x_0 and thus prove that $\alpha_F(x_0) \geqslant n$. Thus the Hölder exponent $\alpha_F(x_0)$ measures how irregular F is at the point x_0. The higher the exponent $\alpha_F(x_0)$, the more regular the function F.

A function F is multi-fractal if $\alpha_F(x)$ differs widely from point to point. In this case, the determination of the Hölder exponents $\alpha_F(x)$ is difficult. Nonetheless the study of such functions has proved important in several domains of physics and signal analysis (for example, see, ref. 1).

The determination of the Hölder exponent of a function can be reduced to estimating its wavelet transform near x_0, using either Proposition 1 or the discrete form of Proposition 1 (see refs. 2–4). Let ψ be a wavelet, i.e., a $C^k(\mathbb{R}^m)$ function where all moments of order less than k vanish and all derivatives of order less than k are well localized (and k is large enough depending on the properties of F we want to analyze). The wavelet transform of F at the position $b \in \mathbb{R}^m$ and for the scale $a > 0$ is

$$C_{a,b}(F) = \frac{1}{a^m} \int_{\mathbb{R}^m} F(t)\, \bar{\psi}\left(\frac{t-b}{a}\right) dt \tag{2}$$

Proposition 1. Let $\alpha < k$.

- $F \in C^\alpha(\mathbb{R}^m)$ if and only if $|C_{a,b}(F)| \leqslant Ca^\alpha$ for all b and sufficiently small a.
- If $F \in C^\alpha(x_0)$, then for sufficiently small a and $|b-x_0| \leqslant 1/2$,

$$|C_{a,b}(F)| \leqslant Ca^\alpha \left(1 + \frac{|b-x_0|}{a}\right)^\alpha \tag{3}$$

- If (3) holds and if $F \in C^\varepsilon(\mathbb{R}^m)$ for an $\varepsilon > 0$, then there exists a polynomial P such that, if $|x-x_0| \leqslant 1/2$,

$$|F(x) - P(x-x_0)| \leqslant C\, |x-x_0|^\alpha \log\left(\frac{2}{|x-x_0|}\right) \tag{4}$$

The pointwise Hölder regularity is summed up by computing the spectrum of singularities $d(\alpha)$ which associates to each α the Hausdorff dimension $d(\alpha)$ of the set E_F^α of points x where $\alpha_F(x) = \alpha$ (conventionally the dimension of the empty set is $-\infty$). A function is called multi-fractal when $d(\alpha)$ is defined at least on an interval of non-empty interior.