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The probability of being decisive

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Abstract. The probability of a voter being decisive (P_D), that is, of one vote affecting the outcome of an election, has generally been incorrectly calculated for the last twenty or more years. The method normally used is due to Banzhaf (1968) and generalised by Beck (1974). It assumes that voters know in advance how many people will vote for each candidate, which is clearly not the case. The correct formulation was given by Good and Mayer in 1975, but was ignored and has subsequently been all but forgotten since then. A simple explanation of these methods is given. Using the incorrect method, errors of magnitude of more than 10^{100} in calculating P_D correctly can be made. The appropriateness of using a decision-theoretic formulation instead of a game-theoretic one is also discussed.

1. Introduction

One of the most important concepts in the theory of voting and public choice is that of decisiveness.[1] The probability of being decisive (P_D) in an election with more than a handful of voters is always small, usually very small, and sometimes infinitesimally small. However, given that the question of who wins, at the margin, is at the heart of social choice theory, the answer to the question of the size of P_D has proved remarkably elusive.

The approach used from the late 1960's to the mid-1980's to decide the issue was based on decision theory, although it was recognised that there were strategic, or game-theoretic, elements, to the problem. That is, in the strategic formulation, voters would be deterred from voting if the result were thought to be a foregone conclusion, as each voter would realise that, individually, his or her vote could not alter the outcome.[2] But if fewer were to vote, the probability that any one voter may be decisive would surely increase, so that in the polar case, with only one person voting, he or she would by definition be decisive. Game-theoretic models,[3] which arose in the 1980's, suggested that multiple equilibria would be possible, if not usual. Since it could not be known in advance which equilibrium would be attained in any particular situation, it is implied that the probability of being decisive is not well-defined in this framework. It would only make sense if probabilities could

be assigned to the occurrence of each of the possible equilibria, and for each of these equilibria, the probability of being decisive determined, given that equilibrium. The overall value of P_D would then be the sum of the products of the respective probabilities for each equilibrium. Other than the simulation work done by Fain and Dworkin (1993), however, there has been no way within these models of discovering the probability with which a particular equilibrium occurs. Thus the question of the size of P_D cannot usefully be answered in a game-theoretic framework.

However, this is patently an exaggeration of the situation which pertains in the real world, as I shall show later in more detail. I shall argue that although there are undoubted game-theoretic considerations involved, we may nevertheless act as though the problem were one of decision theory alone, allowing us to find an answer to the size of P_D.

But here too there is a twist. There are two competing formulae for P_D in the decision-theory formulations. The first, which has been used subsequently, up to the recent past, by a number of scholars in a number of contexts since it was first generalised[4] by Beck (1974), gives a relatively high value of P_D when the result of the election is thought to be knife-edge in closeness, but yields a value which is essentially zero otherwise. The second, proposed by Good and Mayer (1975), is a generalisation[5] of Tullock's (1967) demonstration that P_D should be of the order of N^{-1}, where N is the number in the population who cast a vote. Good and Mayer's method has largely been ignored, forgotten or unknown by most who have needed to calculate the value of P_D, although Chamberlain and Rothschild (1981) apparently independently discovered results which were in accord with those of Good and Mayer.

I shall find it useful to proceed as follows. To begin with, I shall assume that the decision-theoretic model is an appropriate one to use. Given this, I shall show why Good and Mayer's model is the correct one to be employed, in place of Beck's model. It is important to clear up this matter, not only to get the methodology correct, but also because conclusions being drawn using Beck's methodology have been quite misleading at times. I shall make quite explicit that the P_D which is calculated using Good and Mayer's method is a subjective probability based on rational expectations.

Considering a decision-theoretic model first (rather than pitting the decision-theory model against those derived from game theory) gives us insights about the determinants of P_D within this framework. We then use this knowledge to argue that, *in practice*, the conditions under which it is appropriate to use decision theory rather than game theory are close to being met. Thus, at least for the exercise of finding the size of P_D (and possibly for solving some other problems in voting theory) it is appropriate to utilise the