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INTENTIONAL GAPS IN MATHEMATICAL PROOFS

Mind the Gap.

– Announcement made in London tube stations

A central question in the philosophy of mathematics is: How do mathematicians know that mathematical propositions are true? The standard answer is that a mathematician knows that a proposition is true because she knows a proof of that proposition. Of course, this answer leaves a couple of important things unexplained. First, how does the mathematician know that the axioms that her proof appeals to (e.g., the axiom of choice) are true? Second, how does the mathematician know that the rules of inference that her proof appeals to (e.g., the law of the excluded middle) are truth preserving? A lot of work in the philosophy of mathematics has gone into trying to give answers to these two questions (see, e.g., Maddy 1988). In this paper, I argue that even if we had completely satisfactory answers to these two questions, we would still not have a completely satisfactory explanation of how mathematicians actually know *on the basis of proof* that mathematical propositions are true. The reason is that mathematicians often intentionally leave gaps in their proofs.

1. INTRODUCTION

There are a number of ways in which a mathematician might be justified in believing that a mathematical proposition is true. First, working like a scientist, the mathematician might have verified that a number of consequences of the mathematical proposition in question are true. For example, Mark Steiner argues that the mathematician Leonhard Euler knew that $1+1/2^2+1/3^2+\cdots+1/n^2+\cdots = \pi^2/6$ on the basis of inductive evidence of this sort (see Steiner 1975, 102–108). Second, the mathematician might have heard from a reliable source that the mathematical proposition is true. For example, many mathematicians were justified in believing that

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Fermat's Last Theorem was true only because they had read in the *Notices of the American Mathematical Society* that it had been proved.

Even though there are many ways in which a mathematician might be justified in believing that a mathematical proposition is true, there is only one way in which a mathematician feels that she is truly justified. Specifically, the mathematician has to know a proof of the mathematical proposition. According to the logician Gottlob Frege, "it is in the nature of mathematics always to prefer proof, where proof is possible, to any confirmation by induction" (Frege 1980, 2).[1] The mathematician Joseph Shoenfield makes much the same point when he says that "a mathematician may, on occasions, use observation; for example, he may measure the angles of many triangles and conclude that the sum of the angles is always 180°. However, he will accept this as a law of mathematics only when it has been proved" (Shoenfield 1967, 1). In fact, even Euler did not think that he had acceptable mathematical evidence for his belief that $1 + 1/2^2 + 1/3^2 + \cdots + 1/n^2 + \cdots = \pi^2/6$. He said that "we should take great care not to accept as true such properties of the numbers which we have discovered by observation and which are supported by induction alone" (Quoted in Pólya 1954, 3).

The upshot of this rhetoric is that a mathematician has to know a proof of a mathematical proposition in order to be justified *in a manner that is accepted by the mathematical community* in believing that the proposition is true.[2] This leaves us, however, with two further questions: (a) What does a mathematician have to do in order to know a proof? and (b) How does doing so justify her belief that a mathematical proposition is true?

There is a more or less standard account of what a mathematician has to do in order to know a proof and of how doing so justifies her belief. The gist of this story can be found in René Descartes' *Rules for the Direction of the Mind*. According to Descartes, a mathematical proposition must be "deduced from true and known principles by the continuous and uninterrupted action of a mind that has a clear vision of each step in the process" (Descartes 1927, 47). The mathematician has to "trace the whole chain of intermediate conclusions" (Descartes 1927, 62) of the argument. In other words, in order to know a proof of a mathematical proposition, a mathematician has to work through all the details of the proof step by step. Furthermore, if a mathematician has worked through all the details of a proof, then it is fairly clear how she is justified in believing that there is a proof of the proposition and that the proposition is true.

The vast majority of mathematicians agree with Descartes that this is "the only method by which one truly justifies that one knows that A follows from B" (Azzouni 1994, 169). Nevertheless, there has been a lot of contro-