New NCP-Functions and Their Properties

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Abstract. Recently, Luo and Tseng proposed a class of merit functions for the nonlinear complementarity problem (NCP) and showed that it enjoys several interesting properties under some assumptions. In this paper, adopting a similar idea to that of Luo and Tseng, we present new merit functions for the NCP, which can be decomposed into component functions. We show that these merit functions not only share many properties with the one proposed by Luo and Tseng but also enjoy additional favorable properties owing to their decomposable structure. In particular, we present fairly mild conditions under which these merit functions have bounded level sets.

Key Words. Nonlinear complementarity problems, NCP-functions, merit functions, unconstrained optimization reformulations, error bounds, bounded level sets.

1. Introduction

The nonlinear complementarity problem (NCP) is to find a point \( x \in \mathbb{R}^n \) such that

\[
\langle x, F(x) \rangle = 0, \quad x \in \mathbb{R}^n_+, \quad F(x) \in \mathbb{R}^n_+,
\]

where \( F \) is a function from \( \mathbb{R}^n \) into itself, \( \langle \cdot, \cdot \rangle \) denotes the inner product.
in $\mathbb{R}^n$, and $\mathbb{R}^+_n := \{x \in \mathbb{R}^n | x_i \geq 0, i = 1, \ldots, n\}$; see Refs. 1 and 2. Recently, reformulations of the NCP as a minimization problem or a system of equations have drawn much attention (Ref. 3). A function which can constitute an equivalent minimization problem for the NCP is called a merit function. More precisely, a merit function is a function whose global minima on a set $X \subseteq \mathbb{R}^n$ are coincident with the solutions of the original NCP. To construct such a function, it is effective to take advantage of the Cartesian structure of the NCP. In particular, the class of functions defined below serves as a convenient tool for constructing a merit function (Refs. 4-7).

**Definition 1.1.** A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is called an NCP-function if

$$\phi(a, b) = 0 \iff ab = 0, a \geq 0, b \geq 0.$$  

By using an NCP-function $\phi$, we can construct a merit function $g : \mathbb{R}^n \to \mathbb{R}$ for the NCP as follows: If $\phi$ is nonnegative on $\mathbb{R}^2$, then define the function $g$ by

$$g(x) = \sum_{i=1}^{n} \phi(x_i, F_i(x));$$

otherwise, define $g$ by

$$g(x) = \sum_{i=1}^{n} \phi(x_i, F_i(x))^2.$$ 

It is easy to see that the NCP can be cast as the unconstrained optimization problem with the objective function $g$. There are many functions that belong to the class of NCP-functions. Among others, the following three NCP-functions have been well studied in the literature.

**Example 1.1.**

(i) $\phi_{NR}(a, b) = \min\{a, b\}$.

(ii) $\phi_{MS}(a, b) = ab + (1/2a)(\max\{0, a - ab\}^2 - a^2 + \max\{0, b - aa\}^2 - b^2)$, $a > 1$.

(iii) $\phi_{FB}(a, b) = \sqrt{a^2 + b^2} - a - b$.

The merit function based on the function $\phi_{NR}$ is called the natural residual. The function $\phi_{MS}$ is nonnegative on $\mathbb{R}^2$ (Ref. 6), and the merit function based on $\phi_{MS}$ is the implicit Lagrangian proposed by Mangasarian and Solodov (Ref. 8). The function $\phi_{FB}$ was first considered by Fischer (Ref. 9) and attributed to Burmeister. Recently, the merit function based on $\phi_{FB}$ has been extensively studied and has been shown to have a number of favorable properties. Among the above three NCP-functions, the following useful relations hold.