LARGE SCALE SYSTEMS CONTROL

On a Basic Hypothesis of Hierarchical Games Theory

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Abstract—In the study of hierarchical games with a fixed sequence of moves, a common hypothesis is accepted regarding the behavior of a low-level player. It allows for significant simplifying both the solution to the resulting problem and its reasoning. Below we discuss adequacy of this hypothesis. An elementary two-player game serves to illustrate the major problems and some approaches to their solution.

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1. INTRODUCTION

Assume there is a certain game of two persons; the first player chooses his strategies from a set $U$, while the second one selects the strategies from a set $V$. Suppose that the players strive for maximizing the functions $g$ and $h$, respectively.

Consider the following interaction scheme of the players (traditionally, it is referred to as the game $\Gamma_2$ [3]). Imagine that the first player expects to have—and would actually possess—information on the strategy $v \in V$ chosen by his partner (before the first player makes the final decision). Therefore, strategies of the first player are different functions $u^*$ mapping the set $V$ into the set $U$ (denote by $U^*$ the set of all strategies of the first player). We suppose that player 1 selects a strategy $u^* \in U^*$ and then reports it to the partner.

What should be the actions of the first player in the stated conditions? To answer the question formulated, let us involve common methodological principles provided in [4]. For the present work, the following principles are of crucial importance.

Principle 1. Efficiency of any strategy must be assessed on the basis of the guaranteed result gained.

Principle 2. Operations analyst must use all available information.

Therefore, Principle 2 implies that we should employ information on the payoff function of the second player to estimate the set of his feasible responses to the strategy $u_*$. However, how this could be done? As far as the second player knows the partner’s strategy, his results depend only on his actions. Thus, it would be natural to assume that player 2 definitely chooses a certain maximum point of his criterion $h(u_*(v), v)$. At least, two issues are immediate then.

Issue 1. How should the behavior of the second player be described if the function $h(u_*(v), v)$ has no maximum points?

Issue 2. Could we suppose that the second player strives for the exact implementation of his maximum payoff?

Note that Issue 1, par excellence, represents a mathematical problem; consequently, it allows for rather a simple solution. In contrast, Issue 2 most probably appears to be of intensional character. As the result, treating it requires a subtle reasoning.
Starting from the paper [9], the stated problems are solved in the following way. For player 2, the set of rational responses to the strategy \( u^* \) is defined by

\[
B(u^*) = \left\{ v \in V : h(u^*(v), v) = \max_{w \in V} h(u^*(w), w) \right\}
\]

(if the maximum value in formula (1) exists) or by

\[
B(u^*) = \left\{ v \in V : h(u^*(v), v) \geq \sup_{w \in V} h(u^*(w), w) - \delta(u^*) \right\}
\]

(otherwise). Here \( \delta \) is a positive functional given on the set \( U^* \). A player with the described strategy is said to be absolutely rational. Then the strategy \( u^* \) ensures the gain

\[
r(u^*) = \inf_{v \in B(u^*)} h(u^*(v), v),
\]

to the first player. The corresponding guaranteed result of the mentioned player makes

\[
R = \sup_{u^* \in U^*} \inf_{v \in B(u^*)} h(u^*(v), v).
\]

Hence, both issues are answered; we underline that Issue 2 has a definitely positive answer. Meanwhile, this obviously contradicts the first methodological principle above. Indeed, a popular hero of Russian literature, Mr. Plyushkin (depicted by famous Russian writer N.V. Gogol') follows the strategies of hoarding and overstocking; it seems that Mr. Plyushkin exists at the heart of almost everybody. Still, claiming that actual behavior of the people is exactly of Mr. Plyushkin’s type is an idealization. It could be justified only if the resulting solution changes slightly (at least, in common situations). In addition, recall standard arguments that the payoff function of an opponent always includes an uncertainty, that round-off errors are unavoidable for maximum evaluation, etc. (see [1, 11, 12, 15].

On the face of it, such subtilization attracts those investigators focused on mathematical rigor. Nevertheless, a detailed knowledge of hierarchical games indicates their instability against small variations of the model parameters (for instance, see [5, 13]). Thus, the issue under consideration is by no means an idle one.

We emphasize that, following the paper [9], the stated technique was widely adopted by the researchers. Numerous examples and an incomprehensive list of references could be found in the monographs [5, 10]. The issues studied in the present paper are relevant in all these cases. Still, the author attempts to assert that the issues are solved using the same scheme.

2. ATTAINABILITY OF THE MAXIMUM POINT

Consider the first issue formulated above. Here and in the sequel, we suppose that the sets \( U \) and \( V \) possess a topology and are compact, while the functions \( g \) and \( h \) are continuous.

Let us introduce the following assumption.

**Hypothesis 1.** If \( v \in B(u^*) \) and \( h(u^*(w), w) > h(u^*(v), v) \), then \( w \in B(u^*) \).

Evidently, this is the weakest condition matching the idea that the function \( h \) provides (even a poor) description of the second player’s goals.

The hypothesis corresponds, to a certain extent, to the concept of limited rationality suggested by G. Simon in the 1950s [16].

In the case of a game with absolutely rational player 2, Hypothesis 1 holds true.